Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 259 (2010) 2333–2365

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Initial boundary value problem and asymptotic stabilization of the Camassa–Holm equation on an interval

Vincent Perrollaz

Université Pierre et Marie Curie–Paris 6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005, France

Received 4 March 2010; accepted 21 June 2010

Available online 3 July 2010

Communicated by J. Coron

Abstract

We investigate the nonhomogeneous initial boundary value problem for the Camassa–Holm equation on an interval. We provide a local in time existence theorem and a weak-strong uniqueness result. Next we establish a result on the global asymptotic stabilization problem by means of a boundary feedback law.

Keywords: Camassa–Holm equation; Initial boundary value problem; PDE control; Asymptotic stabilization

1. Introduction

1.1. Origins of the equation and presentation of the problems

This article presents results concerning the initial boundary value problem and the possibility of asymptotic stabilization of the Camassa–Holm equation on a compact interval by means of a stationary feedback law acting on the boundary. The Camassa–Holm equation reads as follows (with κ a real constant):

$$\partial_t v - \partial^3_{txx} v + 2\kappa.\partial_x v + 3v.\partial_x v = 2\partial_x v.\partial^2_{xx} v + v.\partial^3_{xxx} v \quad \text{for } (t, x) \in [0, T] \times [0, 1]. \tag{1}$$

E-mail address: perrollaz@ann.jussieu.fr.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.06.007

The Camassa–Holm equation describes one-dimensional surface waves at a free surface of shallow water under the influence of gravity. Here $v(t,x)$ represents the fluid velocity at time t and position x. It is interesting to note that according to [2], it can equally represents the water elevation.

Eq. (1) was first introduced by Fokas and Fuchssteiner [17] as a bi-Hamiltonian model, and was derived later as a water wave model by Camassa and Holm [2]. It turns out that this equation was also obtained as a model for propagating waves in cylindrical elastic rods, see Dai [12].

Eq. (1) shares many features with the KdV equation, see [21]. It is bi-Hamiltonian, completely integrable, and admits soliton solutions see [2,7,9,17,23]. However, it can also model breaking waves, in fact in $H^s(\mathbb{T})$ ($s > \frac{3}{2}$) the solution generally develops singularity in finite time, see [4–6].

The Cauchy problem of (1) has been investigated in great details both on the torus and on the real line, see [1,3,8,13,14,20,24,26]. On the other hand, the study of the initial boundary value problem is much less complete, the homogeneous case was treated in [15] and in a more general setting in [16]. Finally a special case of the inhomogeneous case is considered in [28] (the boundary condition is that there is a constant C such that $\forall t \geqslant 0$ we have $v(t,x) \underset{|x|\to+\infty}{\longrightarrow} C$).

The first part of this article will be devoted to the proofs of a local in time existence theorem and of a weak-strong uniqueness result for the initial boundary value problem of (1).

To explain our boundary formulation of (1), let us first remark that (1) is equivalent to the system:

$$\begin{cases} \partial_t y + v.\partial_x y = -2y.\partial_x v, \\ y - \kappa = \left(1 - \partial_{xx}^2\right)v. \end{cases} \tag{2}$$

This formulation of (1) and the vorticity formulation of the two-dimensional Euler equation for incompressible perfect fluids (U is the speed and ω its vorticity) share similarities:

$$\begin{cases} \partial_t \omega + (U.\nabla)\omega = 0, \\ \operatorname{div} U = 0, \\ \operatorname{curl} U = \omega. \end{cases} \tag{3}$$

In both (2) and (3) there is a coupling between a transport equation and a stationary elliptic one. The initial boundary value problem for the two-dimensional incompressible Euler equation was treated by Yudovitch in [27], where he showed that the problem is well-posed in a classical sense with strong solutions if one prescribes the initial velocity or vorticity, the normal velocity on the boundary and also the vorticity of the fluid on the parts of the boundary where fluid enters.

Similarly we will study the initial boundary value problem of (2) with v prescribed on the boundary, and y prescribed at time 0 and on the parts of the boundary where fluid enters.

Remark 1. Note that (2) is even more similar to the vorticity formulation of the three-dimensional incompressible Euler equation which reads:

$$\begin{cases} \partial_t \omega + (U.\nabla)\omega = (\omega.\nabla)U, \\ \operatorname{div} U = 0, \\ \operatorname{curl} U = \omega \end{cases} \tag{4}$$

because here we have a stretching term $(\omega.\nabla)U$ similar to the term $-2y\,\partial_x v$ in (2). Kazhikov has studied the local in time initial boundary value problem in three dimensions see [22]. However the Euler equation is much less understood in three dimensions. For example it is still unknown whether a singularity may appear in finite time, see [25]. Furthermore the asymptotic stabilization problem is still open for the three-dimensional incompressible Euler equation which is not the case in two dimensions thanks to the papers of Coron [11] and Glass [18].

In the second part of the article we will investigate Eq. (1) from the perspective of control theory. For a general control system

$$\begin{cases} \dot{x} = f(x,u), \\ x(t_0) = x_0 \end{cases} \tag{5}$$

(x being the state of the system and u the so-called control), we can consider two classical problems among others in control theory.

1. First the exact controllability problem which asks, given two states x_0 and x_1 and a time T to find a certain function $u(t)$ such that the solution to (5) satisfies $x(T) = x_1$.
2. If $f(0,0) = 0$, the problem of asymptotic stabilization by a stationary feedback law asks to find a function $u(x)$, such that for any state x_0 a solution $x(t)$ to $\dot{x}(t) = f(x(t), u(x(t)))$, $x(t_0) = x_0$ is global, satisfies $x(t) \underset{t\to+\infty}{\longrightarrow} 0$ and also

$$\forall R > 0,\ \exists r > 0 \text{ such that } \|x_0\| \leqslant r \quad \Rightarrow \quad \forall t \in \mathbb{R},\ \big\|x(t)\big\| \leqslant R. \tag{6}$$

It may seem that if we have controllability, the asymptotic stabilization property is weaker. Indeed for any initial state x_0, we can find T and $u(t)$ such that the solution to (5) satisfies $x(T) = 0$ in this way we stabilize 0 in finite time. However this control suffers from a lack of robustness with respect to perturbation. Indeed with any error on the model, or on the initial state, the state at time T will only be approximately 0. This can be disastrous if $x = 0$ is unstable for the equation $\dot{x} = f(x,0)$. This motivates the problem of asymptotic stabilization by a stationary feedback law which is clearly more robust. In fact in finite dimension, it automatically provides a Lyapunov function.

Concerning the Camassa–Holm equation, O. Glass provided in [19] the first results for the controllability and stabilization. More precisely he considered:

$$\begin{aligned} &\partial_t v - \partial^3_{txx} v + 2\kappa.\partial_x v + 3v.\partial_x v = 2\partial_x v.\partial^2_{xx} v + v.\partial^3_{xxx} v + g(t,x)\mathbf{1}_\omega(x) \\ &\quad \text{for } (t,x) \in [0,T] \times \mathbb{T}, \end{aligned} \tag{7}$$

where the control is the function g, and ω is a nonempty open subset of the torus $\mathbb{T}$. He proved that for any time $T > 0$ we have exact controllability in $H^s(\mathbb{T})$ ($s > \frac{3}{2}$), and also proposed a stationary feedback law $g : H^2(\mathbb{T}) \to H^{-1}(\omega)$ that stabilizes the state $v = -\kappa$ in $H^2(\mathbb{T})$. We will consider those problems, but in our case the control will be the boundary values of v and y. Since $[0,1]$ can be seen as $\mathbb{T} \setminus \omega$ the result of Glass on exact controllability by a distributed term on the torus implies a controllability result by boundary terms as soon as the initial boundary value problem makes sense, which will be the case by the end of the first part of this article (we also need enough regularity on the solution).

Therefore we will only investigate the asymptotic stabilization by a stationary feedback law acting on the boundary of (1). This time again we will consider the analogy with the asymptotic stabilization of the two-dimensional Euler equation of incompressible fluids results by Coron [11] for a simply connected domain and Glass [18] for a general domain. It should be remarked that in three dimensions the problem of asymptotic stabilization is still open. In both cases one of the main difficulty is that the linearized system around the equilibriums (which are $(y, v) = (0, -\kappa)$ for (2) and $(\omega, U) = (0, 0)$ for (3)) is not stabilizable, so we will use the so-called return method introduced by Coron in [10]. Since the evolution equation of (2) is on y, it will be much easier to work if we consider y and not v to be the state of the system.

1.2. Results

We begin with a general remark that will be used many times later.

Remark 2. Changing $v(t, x)$ in $-v(t, 1 - x)$ and $y(t, x)$ in $-y(t, 1 - x)$ we change κ into $-\kappa$, therefore from now on we will suppose that $\kappa \leqslant 0$ (this choice is more convenient for the stabilization part).

Let T be a positive number. In the following we take $\Omega_T = [0, T] \times [0, 1]$. Let v_l and v_r be in $\mathcal{C}^0([0, T], \mathbb{R})$ and $y_0 \in L^\infty(0, 1)$. We set

$$\Gamma_l = \big\{t \in [0, T] \,\big|\, v_l(t) > 0\big\} \quad \text{and} \quad \Gamma_r = \big\{t \in [0, T] \,\big|\, v_r(t) < 0\big\}.$$

In the following, we will always suppose that the sets

$$P_l = \big\{t \in [0, T] \,\big|\, v_l(t) = 0\big\} \quad \text{and} \quad P_r = \big\{t \in [0, T] \,\big|\, v_r(t) = 0\big\} \tag{8}$$

have a finite number of connected components. Finally let $y_l \in L^\infty(\Gamma_l)$ and $y_r \in L^\infty(\Gamma_r)$. The functions v_l, v_r, y_l and y_r will be the boundary values for the equation and y_0 is the initial data.

Let now $\mathcal{A}$ be the auxiliary function which lifts the boundary values v_l and v_r and is defined by:

$$\begin{cases} \big(1 - \partial_{xx}^2\big)\mathcal{A}(t, x) = 0, & \forall (t, x) \in \Omega_T, \\ \mathcal{A}(t, 0) = v_l(t), \quad \mathcal{A}(t, 1) = v_r(t), & \forall t \in [0, T]. \end{cases} \tag{9}$$

Setting $v = u + \mathcal{A}$, we can further rewrite the system (2) as:

$$\begin{cases} y(t, x) - \kappa = \big(1 - \partial_{xx}^2\big)u(t, x), \ dx, \\ u(t, 0) = u(t, 1) = 0, \ dt \text{ a.e.}, \end{cases} \tag{10}$$

$$\begin{cases} \partial_t y + (u + \mathcal{A}).\partial_x y = -2y.\partial_x(u + \mathcal{A}), \\ y(0, .) = y_0, \quad y(., 0)_{|\Gamma_l} = y_l \quad \text{and} \quad y(., 1)_{|\Gamma_r} = y_r. \end{cases} \tag{11}$$

The meaning of being a solution to (10)–(11) will be specified later but we will have $u \in L^\infty((0, T); \mathrm{Lip}([0, 1]))$ and $y \in L^\infty(\Omega_T)$. In the first part of this article, we will be interested in the initial boundary value problem on the interval for the system (10)–(11). We will first prove a local in time existence theorem:

Theorem 1. *For $\tilde{T} > 0$, we consider $v_l, v_r \in \mathcal{C}^0([0,\tilde{T}])$ such that the sets P_l and P_r have only a finite number of connected components. Let $y_0 \in L^\infty(0,1)$, $y_l \in L^\infty(\Gamma_l)$ and $y_r \in L^\infty(\Gamma_r)$. There exist $T > 0$, and (u, y) a weak solution of the system* (10)–(11) *with $u \in L^\infty((0,T); \mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}([0,T]; H_0^1(0,1))$ and $y \in L^\infty(\Omega_T)$. Moreover any such solution u is in fact in $\mathcal{C}^0([0,T]; W^{2,p}(0,1)) \cap C^1([0,1]; W_0^{1,p}(0,1))$, $\forall p < +\infty$. Furthermore the existence time of a maximal solution is larger than $\min(\tilde{T}, T^*)$, with*

$$T^* = \max_{\beta>0}\left(\frac{\ln(1+\beta/C_0)}{2(C_1 + (2+\sinh(1))(C_0 + |\kappa| + \beta))}\right), \tag{12}$$

$$C_0 = \max\bigl(\|y_0\|_{L^\infty(0,1)}, \|y_l\|_{L^\infty(\Gamma_l)}, \|y_r\|_{L^\infty(\Gamma_l)}\bigr), \tag{13}$$

$$C_1 = \frac{1}{\tanh(1)}.\bigl(\|v_r\|_{L^\infty(0,T)} + \|v_l\|_{L^\infty(0,T)}\bigr). \tag{14}$$

In a second step, we will show a weak-strong uniqueness property:

Theorem 2. *Let $(u, y) \in L^\infty((0,T); \mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}([0,T]; H_0^1(0,1)) \times L^\infty([0,T]; \mathrm{Lip}([0,1]))$ be a weak solution of* (10) *and* (11) *then it is unique in $L^\infty((0,T); \mathcal{C}^{1,1}([0,1])) \times L^\infty(\Omega_T)$.*

In the second part of the paper, we will be interested in the asymptotic stabilization of the system (1) by a boundary feedback law. Let $A_l > 2.\sinh(1)$, $A_r > A_l.\cosh(1) + \sinh(2)$, $M > 0$ and $T > 0$. Our feedback law for (2) reads:

$$y \in \mathcal{C}^0\bigl([0,1]\bigr) \mapsto \begin{cases} v_l(y) = A_l.\|y\|_{\mathcal{C}^0([0,1])} - \kappa, \\ v_r(y) = A_r.\|y\|_{\mathcal{C}^0([0,1])} - \kappa, \\ \dot{y}_l(t) + M.y_l(t) = 0. \end{cases} \tag{15}$$

This allows us to get the following theorem:

Theorem 3. *For any $y_0 \in \mathcal{C}^0([0,1])$ there exists $(y, v) \in \mathcal{C}^0(\Omega_T) \times \mathcal{C}^0([0,T], \mathcal{C}^2([0,1]))$ a weak solution of* (2) *and* (15) *satisfying*

$$\forall x \in [0,1], \quad y(0,x) = y_0(x). \tag{16}$$

Furthermore any maximal solution of (2), (15) *and* (16) *is global, and if we let*

$$c = \min\left(A_l - 2.\sinh(1), \frac{A_r - A_l.\cosh(1) - \sinh(2)}{\sinh(1)}\right) \quad \text{and} \quad \tau = \frac{1}{M}.\ln\left(\frac{2.c.\|y_0\|_{\mathcal{C}^0([0,1])}}{M}\right)$$

then we have:

$$\forall t \geqslant \tau \quad \bigl\|y(t,.)\bigr\|_{\mathcal{C}^0([0,1])} \leqslant \frac{M}{2c}.\frac{1}{1+M(t-\tau)}.$$

2. Initial boundary value problem

We first define what we mean by a weak solution to (11). Our test functions will be in the space:

$$Adm(\Omega_T) = \{\psi \in \mathcal{C}^1(\Omega_T) \mid \psi(t,x) = 0 \text{ on } [0,T] \setminus \Gamma_l \\ \times \{0\} \cup [0,T] \setminus \Gamma_r \times \{0\} \cup \{T\} \times [0,1]\}. \tag{17}$$

Definition 1. When $u \in L^\infty((0,T); \mathrm{Lip}([0,1]))$, a function $y \in L^\infty(\Omega_T)$ is a weak solution to (11) if $\forall \psi \in Adm(\Omega_T)$:

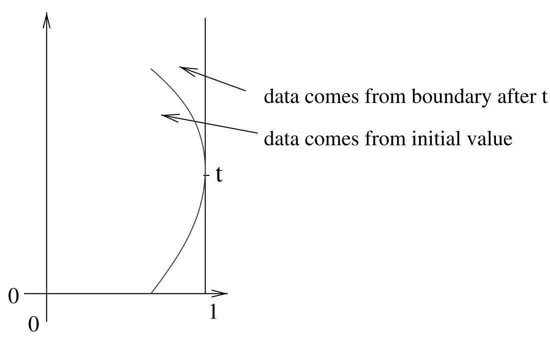
$$\iint_{\Omega_T} y\big(\partial_t \psi + (u+\mathcal{A})\partial_x \psi - \partial_x(u+\mathcal{A})\psi\big)\, dt\, dx \\ = -\int_0^1 y_0(x)\psi(0,x)\, dx + \int_0^T \big(\psi(t,1)v_r(t)y_r(t) - \psi(t,0)v_l(t)y_l(t)\big)\, dt.$$

Remark 3. It is obvious that $\mathcal{C}_0^1(\Omega_T) \subset Adm(\Omega_T)$ therefore a weak solution to (11) is also a solution to (11) in the distribution sense. And it is then clear that a regular weak solution is a classical solution.

2.1. Strategy

In this part we will prove Theorems 1 and 2. Let us first explain the general strategy.

We want to solve (10) and (11). Eq. (10) is a linear elliptic equation, and with u fixed (11) is a linear transport equation in y, with boundary data. Even when the flow is regular enough (and it will be in our case) to use the method of characteristics to solve the equation, singularity will generally appear, no matter how smooth the initial and boundary datas are, because of the boundary.



It is therefore useful to deal with weak solution of (11) belonging to $L^\infty(\Omega_T)$. This is done in Appendix A. Once we know how to deal with each equation separately and have appropriate linear estimates, we use a fixed point strategy. It is interesting to remark that Yudovitch dealt with the two-dimensional incompressible Euler equation with nonhomogeneous boundary conditions in a similar way. However with y only essentially bounded, we cannot easily estimate the difference of two couples (u_1, y_1) and (u_2, y_2), therefore we will rather use a compactness argument and a Schauder fixed point instead of a Banach fixed point. The auxiliary function $\mathcal{A}$ may be less regular in time than u and this is why we will be able to transfer the time regularity of y on u. We will only prove a weak-strong uniqueness property, for the same reason that prevented us from using a Banach fixed point theorem.

Therefore in Section 2.2 we will define precisely the fixed point operator $\mathcal{F}$ and study some of its properties. In Section 2.3 we will precise the domain on which we will apply Schauder's fixed point theorem, we will prove the continuity of $\mathcal{F}$ in Section 2.4 and also study the additional properties of a fixed point. Finally in Section 2.5 we will prove the weak-strong uniqueness property.

2.2. The operator $\mathcal{F}$

The operator $\mathcal{F}$ is obtained as follows. Given u in $L^\infty((0,T);\mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}([0,T];H_0^1(0,1))$ we will define y to be the solution of (11), and once we have y in $L^\infty(\Omega_T)$, we introduce $\tilde{u}$ solution of

$$\big(1 - \partial_{xx}^2\big)\tilde{u} = y - \kappa. \tag{18}$$

Then $\mathcal{F}$ is defined as the operator associating $\tilde{u}$ to u.

Now let us describe the auxiliary function $\mathcal{A}$ once and for all.

Proposition 2.1. *The function $\mathcal{A}$ defined by* (9) *satisfies*:

$$\forall (t,x) \in \Omega_T \quad \mathcal{A}(t,x) = \frac{1}{\sinh(1)}.\big(\sinh(x).v_r(t) + \sinh(1-x).v_l(t)\big),$$
$$\mathcal{A} \in \mathcal{C}^0\big([0,T];\mathcal{C}^\infty\big([0,1]\big)\big), \quad \textit{and hence}$$
$$\|\mathcal{A}\|_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant \frac{\cosh(1)}{\sinh(1)}.\big(\|v_r\|_{L^\infty(0,T)} + \|v_l\|_{L^\infty(0,T)}\big).$$

As in Section A.1, for a function $u \in L^\infty((0,T);\mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}([0,T];H_0^1(0,1))$ we consider ϕ the flow of $u + \mathcal{A}$. For $(t,x) \in \Omega_T$, $\phi(.,t,x)$ is defined on a set $[e(t,x), h(t,x)]$, here $e(t,x)$ is basically the entrance time in Ω_T of the characteristic curve going through (t,x).

Lemma 1. *The flow ϕ satisfies the following properties*:

1. *ϕ is $\mathcal{C}^1$ with the following partial derivatives*

$$\partial_1\phi(s,t,x) = (u+\mathcal{A})\big(s,\phi(s,t,x)\big),$$
$$\partial_2\phi(s,t,x) = -(u+\mathcal{A})(t,x).\exp\left(\int_t^s \partial_x(u+\mathcal{A})\big(r,\phi(r,t,x)\big)\,dr\right),$$

$$\partial_3\phi(s,t,x) = \exp\left(\int_t^s \partial_x(u+\mathcal{A})\big(r,\phi(r,t,x)\big)\,dr\right),$$

2. $\forall j \in \{1,2,3\}$, $\|\partial_j\phi\|_{\mathcal{C}^0} \leqslant (1+\|u+\mathcal{A}\|_{\mathcal{C}^0(\Omega_T)})e^{T.\|\partial_x(u+\mathcal{A})\|_{\mathcal{C}^0(\Omega_T)}}$,
3. *if* $e(t,x) > 0$ *then* $\phi(e(t,x),t,x) \in \{0,1\}$,
4. *if* $h(t,x) < T$ *then* $\phi(h(t,x),t,x) \in \{0,1\}$.

We introduce a partition of Ω_T, which allows us to distinguish the different influence zones in Ω_T.

Definition 2. Let

- $P = \{(t,x) \in \Omega_T \mid \exists s \in [e(t,x),h(t,x)]$ for which $(\phi(s,t,x)=0$ and $v_l(s)=0)$ or $(\phi(s,t,x)=1$ and $v_r(s)=0)\} \cup \{\phi(s,0,0) \mid s \leqslant h(0,0)\} \cup \{\phi(s,0,1) \mid s \leqslant h(0,1)\}$,
- $I = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) = 0\}$,
- $L = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) > 0$ and $\phi(e(t,x)t,x) = 0\}$,
- $R = \{(t,x) \in \Omega_T \setminus P \mid e(t,x) > 0$ and $\phi(e(t,x)t,x) = 1\}$.

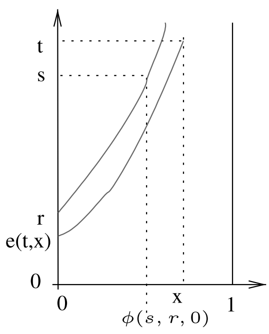
Remark 4. The set P is constituted of the problematic points. Indeed those points belong to the characteristics tangent to the boundary, which are precisely the singular points of e and h.

Proposition 2.2. *We have the following properties.*

1. *The sets P, I, L and R constitute a partition of Ω_T.*
2. *The set P is negligible and each spatial section of P is negligible for the* 1*d lebesgue measure.*
3. *The function e is $\mathcal{C}^1$ on $L \cup R \cup I$.*
4. *If $(t,x) \in L$ then $e(t,x) \in \Gamma_l$ and if $(t,x) \in R$ then $e(t,x) \in \Gamma_r$.*
5. *All those sets are invariant by the flow ϕ.*
6. *If $(t,x) \in L$ then $\forall \tilde{x} \in [0,x]$, $(t,\tilde{x}) \in P \cup L$, if $(t,x) \in R$ then $\forall \tilde{x} \in [x,1]$, $(t,\tilde{x}) \in P \cup R$ and if $(t,x) \in I$ and $(t,x+x') \in I$ then $\forall \tilde{x} \in [x,x+x']$, $(t,\tilde{x}) \in P \cup I$.*

Proof. The points 1, 4, 5, 6 are easy. The second point is true because for any $t \in [0,T]$ the set $\{(t,x) \mid x \in [0,1]\} \cap P\}$ is injected in the set of connected components of P_l and P_r, so it is countable and therefore 1d negligible. It implies that P itself is 2d negligible.

And the third point is shown in Proposition A.3.



□

For $u \in L^\infty((0,T);\mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}([0,T];H^1_0(0,1))$, we define $y \in L^\infty(\Omega_T)$ by:

- if $(t,x) \in I$, $y(t,x) = y_0(\phi(0,t,x)).\exp(-2\int_0^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))\,ds)$,
- if $(t,x) \in L$, $y(t,x) = y_l(e(t,x)).\exp(-2\int_{e(t,x)}^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))\,ds)$,
- if $(t,x) \in R$, $y(t,x) = y_r(e(t,x)).\exp(-2\int_{e(t,x)}^t \partial_x(u+\mathcal{A})(s,\phi(s,t,x))\,ds)$.

And we have:

1. the function y is the unique weak solution of (11) in the sense of Definition 1, thanks to Theorem 6 and Proposition A.7 (which can be applied because $u \in \mathcal{C}^0(\Omega_T)$ and $\partial_x u \in \mathcal{C}^0(\Omega_T)$),
2. since $y \in L^\infty(\Omega_T)$ and satisfies (11), we immediately get $y \in W^{1,\infty}(0,T,H^{-1}(0,1))$,
3. the function y satisfies the estimates:

$$\begin{aligned}\|y\|_{L^\infty(\Omega_T)} &\leqslant \max\big(\|y_0\|_{L^\infty}, \|y_l\|_{L^\infty}, \|y_r\|_{L^\infty}\big)\\ &\qquad \times \exp\big(2T\big(\|\partial_x u\|_{L^\infty(\Omega_T)} + \|\partial_x \mathcal{A}\|_{L^\infty(\Omega_T)}\big)\big), \qquad (19)\\ \|\partial_t y\|_{L^\infty((0,T),H^{-1})} &\leqslant 3.\max\big(\|y_0\|_{L^\infty(0,1)}, \|y_l\|_{L^\infty(\Gamma_l)}, \|y_r\|_{L^\infty(\Gamma_l)}\big)\\ &\qquad \times \exp\big(2T\big(\|\partial_x u\|_{L^\infty(\Omega_T)} + \|\partial_x \mathcal{A}\|_{L^\infty(\Omega_T)}\big)\big)\\ &\qquad \times \big(\|u\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))} + \|\mathcal{A}\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}\big), \qquad (20)\end{aligned}$$

4. if $(t,x) \in I \cup L \cup R$ and if $(s,s') \in [e(t,x),h(t,x)]^2$, one has the following property:

$$y\big(s,\phi(s,t,x)\big) = y\big(s',\phi(s',t,x)\big).\exp\left(-2\int_{s'}^{s} \partial_x(u+\mathcal{A})\big(r,\phi(r,t,x)\big)\,dr\right).$$

We can now focus on the elliptic equation (10).

Lemma 2. *There exists a unique* $\tilde{u} \in L^\infty((0,T),H^1_0(0,1))$ *such that*

$$\forall t \in (0,T), \quad y(t,.) - \kappa = \big(1 - \partial^2_{xx}\big)\tilde{u}(t,.) \quad \text{in } \mathcal{D}'(0,1).$$

Furthermore $\tilde{u} \in L^\infty((0,T);\mathcal{C}^{1,1}([0,1])) \cap \mathrm{Lip}((0,T),H^1_0(0,1))$ *since* $y \in L^\infty(\Omega_T) \cap \mathrm{Lip}([0,T];H^{-1}(0,1))$. *Moreover we have the bounds*

$$\|\tilde{u}\|_{L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant \big(1+2\sinh(1)\big).\big(|\kappa| + \|y\|_{L^\infty(\Omega_T)}\big), \qquad (21)$$

$$\|\partial_t \tilde{u}\|_{L^\infty((0,T);H^1_0(0,1))} \leqslant \|\partial_t y\|_{L^\infty((0,T),H^{-1}(0,1))}. \qquad (22)$$

Proof. In the first point, the constant comes from:

$$\tilde{u}(t,x) = \int_0^x \sinh(x-\tilde{x}).\big(\kappa - y(t,\tilde{x})\big)\,d\tilde{x} - \frac{\sinh(x)}{\sinh(1)}.\int_0^1 \sinh(\tilde{x}).\big(\kappa - y(t,\tilde{x})\big)\,d\tilde{x}. \qquad (23)$$

The second point is classical □

Finally we can define $\mathcal{F}$ by:

$$\forall u \in L^{\infty}\big((0,T);\mathcal{C}^{1,1}\big([0,1]\big)\big) \cap \mathrm{Lip}\big([0,T];\, H_0^1(0,1)\big),$$
$$\mathcal{F}(u) = \tilde{u} \in L^{\infty}\big((0,T);\mathcal{C}^{1,1}([0,1])\big) \cap \mathrm{Lip}\big([0,T];\, H_0^1(0,1)\big). \tag{24}$$

We now introduce a domain for the operator $\mathcal{F}$.

2.3. The domain

Let B_0 and B_1 be positive numbers, then we set:

$$\mathcal{C}_{B_0,B_1,T} = \big\{u \in L^{\infty}\big((0,T);\mathcal{C}^{1,1}([0,1])\big) \cap \mathrm{Lip}\big([0,T];\, H_0^1(0,1)\big) \;\big|\; \text{such that both}$$
$$\|u\|_{L^{\infty}((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant B_0 \text{ and } \|u\|_{\mathrm{Lip}([0,T];\, H_0^1(0,1))} \leqslant B_1\big\}. \tag{25}$$

Obviously $\mathcal{C}_{B_0,B_1,T}$ is convex. We will endow $\mathcal{C}_{B_0,B_1,T}$ with the norm $\|.\|_{L^{\infty}((0,T);\mathrm{Lip}([0,1]))}$.

Lemma 3. *There exist positive numbers B_0, B_1, T, such that $\mathcal{F}$ maps $\mathcal{C}_{B_0,B_1,T}$ into itself.*

Proof. Let us first introduce the two following constants depending only on the initial and boundary conditions

$$C_0 = \max\big(\|y_0\|_{L^{\infty}(0,1)}, \|y_l\|_{L^{\infty}(\Gamma_l)}, \|y_r\|_{L^{\infty}(\Gamma_r)}\big),$$
$$C_1 = \frac{\cosh(1)}{\sinh(1)}.\big(\|v_r\|_{L^{\infty}(0,T)} + \|v_l\|_{L^{\infty}(0,T)}\big).$$

Estimates (19), (20), (21) and (22) on y and $\tilde{u}$ now read:

$$\|y\|_{L^{\infty}(\Omega_T)} \leqslant C_0.\exp\big(2T\big(\|\partial_x u\|_{L^{\infty}(\Omega_T)} + C_1\big)\big),$$
$$\|\tilde{u}\|_{L^{\infty}((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant \big(1 + 2\sinh(1)\big).\big(|\kappa| + \|y\|_{L^{\infty}(\Omega_T)}\big),$$
$$\|\partial_t y\|_{L^{\infty}((0,T);\,H^{-1}(0,1))} \leqslant 3.C_0.\exp\big(2T\big(\|\partial_x u\|_{L^{\infty}(\Omega_T)} + C_1\big)\big).\big(\|u\|_{L^{\infty}((0,T);\mathrm{Lip}([0,1]))} + C_1\big),$$
$$\|\partial_t \tilde{u}\|_{L^{\infty}((0,T);\,H_0^1(0,1))} \leqslant \|\partial_t y\|_{L^{\infty}((0,T);\,H^{-1}(0,1))}.$$

Combining those estimates we get:

$$\|\tilde{u}\|_{L^{\infty}((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant \big(1 + 2\sinh(1)\big).\big(|\kappa| + C_0.\exp\big(2T\big(\|\partial_x u\|_{L^{\infty}(\Omega_T)} + C_1\big)\big)\big),$$
$$\|\partial_t \tilde{u}\|_{L^{\infty}((0,T);\,H_0^1(0,1))} \leqslant 3.C_0.\exp\big(2T\big(\|\partial_x u\|_{L^{\infty}(\Omega_T)} + C_1\big)\big).\big(\|u\|_{L^{\infty}((0,T);\mathrm{Lip}([0,1]))} + C_1\big).$$

Now if $u \in \mathcal{C}_{B_0,B_1,T}$ we have

$$\|\tilde{u}\|_{L^{\infty}((0,T);\mathcal{C}^{1,1}([0,1]))} \leqslant \big(1 + 2\sinh(1)\big).\big(|\kappa| + C_0.\exp\big(2T(B_0 + C_1)\big)\big),$$
$$\|\partial_t \tilde{u}\|_{L^{\infty}((0,T);\,H_0^1(0,1))} \leqslant 3.C_0.\exp\big(2T(B_0 + C_1)\big).(B_0 + C_1).$$

Finally, to obtain $\tilde{u} \in \mathcal{C}_{B_0,B_1,T}$ it is sufficient that

$$\bigl(1+2\sinh(1)\bigr).\bigl(|\kappa|+C_0.\exp\bigl(2T(B_0+C_1)\bigr)\bigr) \leqslant B_0 \quad \text{and}$$
$$B_0+3.C_0.\exp\bigl(2T(B_0+C_1)\bigr).(B_0+C_1) \leqslant B_1.$$

Once we have chosen T and B_0, it is easy to choose B_1 to satisfy the second inequality. For the first one we just choose B_0 sufficiently large and then T close to 0. More precisely:

$$B_0 > \bigl(1+2\sinh(1)\bigr).\bigl(|\kappa|+C_0\bigr),$$
$$T \leqslant \frac{\ln(\frac{B_0}{1+2\sinh(1)}-|\kappa|)-\ln(C_0)}{2(B_0+C_1)}.$$

It only to remains to maximize the bound of T to get the minimum existence, and with $\frac{B_0}{1+2\sinh(1)} = |\kappa|+C_0+\beta$ we get the result announced. □

Let us now prove the compactness of the domain.

Proposition 2.3. *$\mathcal{C}_{B_0,B_1,T}$ is compact with respect to the norm $\|.\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$.*

Proof. The fact that $\mathcal{C}_{B_0,B_1,T}$ is closed in $L^\infty((0,T);\mathrm{Lip}([0,1]))$ follows from the weak* compactness of the domain in $L^\infty((0,T);\mathcal{C}^{1,1}([0,1]))$ and in $\mathrm{Lip}([0,T];H_0^1(0,1))$, and a classical use of a limit uniqueness.

We now show the relative compactness of $\mathcal{C}_{B_0,B_1,T}$ in $L^\infty((0,T);\mathrm{Lip}([0,1]))$. Let (u_n) be a sequence of $\mathcal{C}_{B_0,B_1,T}$. Since $H_0^1(0,1) \hookrightarrow \mathcal{C}^{\frac{1}{2}}([0,1])$ we can extract by Ascoli's theorem a subsequence $(u_{n'})$ converging in $L^\infty(\Omega_T)$. But since we have

$$\forall u \in L^\infty\bigl((0,T);W^{2,\infty}(0,1)\bigr),\quad \|\partial_x u\|_{L^\infty(\Omega_T)} \leqslant 2.\sqrt{\|u\|_{L^\infty(\Omega_T)}.\bigl\|\partial_{xx}^2 u\bigr\|_{L^\infty(\Omega_T)}},$$

we can conclude that $(u_{n'})$ actually converges in $L^\infty((0,T);\mathrm{Lip}([0,1]))$. □

Before applying Schauder's fixed point theorem, it only remains to prove the continuity of the operator $\mathcal{F}$.

2.4. Continuity of $\mathcal{F}$ and properties of the fixed points

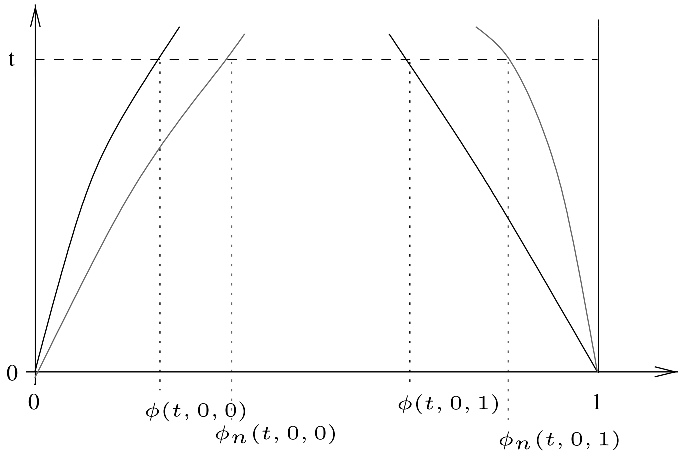
We begin with a result about the continuity of $\mathcal{F}$.

Proposition 2.4. *The operator $\mathcal{F}: \mathcal{C}_{B_0,B_1,T} \to \mathcal{C}_{B_0,B_1,T}$ is continuous with respect to $\|.\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$.*

Proof. Let us take a sequence (u_n) which tends to u with respect to $\|.\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. We call $\tilde{u}_n = \mathcal{F}(u_n)$ and $\tilde{u} = \mathcal{F}(u)$. Denote by ϕ_n the flow of $u_n + \mathcal{A}$ and ϕ the flow of

$u + \mathcal{A}$. Thanks to Proposition A.4, we have that $\phi_n \underset{n\to+\infty}{\longrightarrow} \phi$ locally in $\mathcal{C}^1$. Let us show first that $\|y_n(t,.) - y(t,.)\|_{L^1(0,1)} \underset{n\to 0}{\longrightarrow} 0$ dt a.e.

Let $t \in [0,T]$, having supposed that P_l and P_r have only a finite number of connected components (see (8)), we can assume, reducing t if necessary that v_l and v_r do not change sign on $[0,t]$. We will focus on the case where $v_l \geqslant 0$ and $v_r \leqslant 0$, the situation:



The characteristics of ϕ_n and ϕ may or may not cross before time t, but we are only interested in their relative positions at time t, which here correspond to $\phi(t,0,0) \leqslant \phi_n(t,0,0) \leqslant \phi(t,0,1) \leqslant \phi_n(t,0,1)$. The other cases are proved in the same way. We first point out that since $u_n \in \mathcal{C}_{B_0,B_1,T}$ we have a bound for (y_n) in $L^\infty(\Omega_T)$. Now

$$\begin{aligned}
\int_0^1 \left|y(t,x) - y_n(t,x)\right| dx &= \int_0^{\phi(t,0,0)} \left|y(t,x) - y_n(t,x)\right| dx + \int_{\phi(t,0,0)}^{\phi_n(t,0,0)} \left|y(t,x) - y_n(t,x)\right| dx \\
&\quad + \int_{\phi_n(t,0,0)}^{\phi(t,0,1)} \left|y(t,x) - y_n(t,x)\right| dx + \int_{\phi(t,0,1)}^{\phi_n(t,0,1)} \left|y(t,x) - y_n(t,x)\right| dx \\
&\quad + \int_{\phi_n(t,0,1)}^{1} \left|y(t,x) - y_n(t,x)\right| dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Since $\phi_n(t,0,0) \underset{n\to+\infty}{\longrightarrow} \phi(t,0,0)$ and $\phi_n(t,0,1) \underset{n\to+\infty}{\longrightarrow} \phi(t,0,1)$ and thanks to the uniform bound on $\|y_n\|_{L^\infty(\Omega_T)}$ we see that both I_2 and I_4 tend to 0 when n goes to infinity.

For I_1 we have:

$$I_1 = \int_0^{\phi(t,0,0)} \left| y_l\big(e_n(t,x)\big).\exp\left(-2\int_{e_n(t,x)}^{t} \partial_x(u_n+\mathcal{A})\big(r,\phi_n(r,t,x)\big)\,dr\right) \right.$$
$$\left. - y_l\big(e(t,x)\big).\exp\left(-2\int_{e(t,x)}^{t} \partial_x(u+\mathcal{A})\big(r,\phi(r,t,x)\big)\,dr\right)\right| dx.$$

But thanks to Proposition A.2, if $(t,x)\notin P$ (defined by ϕ) we have $e_n(t,x) \underset{n\to+\infty}{\longrightarrow} e(t,x)$. This implies that if y_l were continuous, since we have a uniform bound on $\|u_n\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$ the dominated convergence theorem would provide:

$$I_1 = \int_0^{\phi(t,0,0)} \big|y(t,x)-y_n(t,x)\big|\,dx \underset{n\to+\infty}{\longrightarrow} 0.$$

The same idea can be applied to I_3 and I_5.

Hence for y_l, y_r and y_0 continuous we have $\|y_n(t,.)-y(t,.)\|_{L^1(0,1)} \underset{n\to+\infty}{\longrightarrow} 0$.

But now thanks to inequality (56), we have:

$$\begin{aligned}\big\|y(t,.)\big\|_{L^1(0,1)} &\leqslant \big(\|y_0\|_{L^1(0,1)}+\|y_l\|_{L^1((0,t)\cap\Gamma_l)}+\|y_r\|_{L^1((0,t)\cap\Gamma_r)}\big)\\ &\quad\times\|u+\mathcal{A}\|_{L^\infty(\Omega_T)}.e^{3t.\|\partial_x(u+\mathcal{A})\|_{L^\infty(\Omega_T)}}, \qquad (26)\\ \big\|y_n(t,.)\big\|_{L^1(0,1)} &\leqslant \big(\|y_0\|_{L^1(0,1)}+\|y_l\|_{L^1((0,t)\cap\Gamma_l)}+\|y_r\|_{L^1((0,t)\cap\Gamma-r)}\big)\\ &\quad\times\|u_n+\mathcal{A}\|_{L^\infty(\Omega_T)}.e^{3t.\|\partial_x(u_n+\mathcal{A})\|_{L^\infty(\Omega_T)}}. \qquad (27)\end{aligned}$$

So by density of $\mathcal{C}^0$ in L^1, and with the uniform bound on $\|u_n\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$, the general case follows,

$$\big\|y_n(t,.)-y(t,.)\big\|_{L^1(0,1)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Now only the restriction on t remains, we recall that until now we supposed that v_l and v_l did not change sign on $[0,t]$.

But if v_l and v_r do not change sign on $[0,t_1]$ and then on $[t_1,t]$, we have $\|y_n(t_1,.)-y(t_1,.)\|_{L^1(0,1)} \underset{n\to+\infty}{\longrightarrow} 0$. Let us call $\tilde{y}_n$ the solution of $\partial_t\tilde{y}_n+(u_n+\mathcal{A})\partial_x\tilde{y}_n = -2.\tilde{y}_n.\partial_x(u_n+\mathcal{A})$ on $[t_1,t]\times[0,1]$ with initial value $y(t_1,.)$ and boundary values y_l, y_r. Due to what precedes we have $\|\tilde{y}_n(t,.)-y(t,.)\|_{L^1(0,1)} \underset{n\to+\infty}{\longrightarrow} 0$. Now we can conclude that:

$$\begin{aligned}&\big\|y_n(t,.)-y(t,.)\big\|_{L^1(0,1)}\\ &\quad\leqslant \big\|y_n(t,.)-\tilde{y}_n(t,.)\big\|_{L^1(0,1)}+\big\|\tilde{y}_n(t,.)-y(t,.)\big\|_{L^1(0,1)}\\ &\quad\leqslant \big\|y_n(t_1,.)-\tilde{y}_n(t_1,.)\big\|_{L^1(0,1)}.\|u_n+\mathcal{A}\|_{L^\infty(\Omega_T)}.e^{3(t-t_1)\|\partial_x(u_n+\mathcal{A})\|_{L^\infty(\Omega_T)}}\\ &\qquad+\big\|\tilde{y}_n(t,.)-y(t,.)\big\|_{L^1(0,1)}\end{aligned}$$

$$\begin{aligned}
&\leqslant \big\| y_n(t_1,.) - y(t_1,.)\big\|_{L^1(0,1)}.\|u_n + \mathcal{A}\|_{L^\infty(\Omega_T)}.e^{3(t-t_1)\|\partial_x(u_n+\mathcal{A})\|_{L^\infty(\Omega_T)}} \\
&\quad + \big\|\tilde{y}_n(t,.) - y(t,.)\big\|_{L^1(0,1)} \\
&\underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore the convergence in $L^1(0,1)$ propagates on each interval where v_l and v_r do not change sign, thanks to the hypothesis on P_r and P_l we have:

$$\forall t \in [0,T] \quad \big\| y_n(t,.) - y(t,.)\big\|_{L^1(0,1)} \underset{n\to+\infty}{\longrightarrow} 0. \tag{28}$$

Combining this first convergence result with the uniform bound of $y_n - y$ in $L^\infty(\Omega_T)$ and using the dominated convergence theorem in the time variable we obtain:

$$y_n \to y \quad \text{in } L^1(\Omega_T).$$

In term of $\tilde{u}$ and $\tilde{u}_n$ it implies that

$$\tilde{u}_n \to \tilde{u} \quad \text{in } L^1\big(0,T,W^{2,1}(0,1)\big).$$

But we also have $\forall n \in \mathbb{N}$ $\mathcal{F}(u_n) \in \mathcal{C}_{B_0,B_1,T}$, and we know (see 2.3) that $\mathcal{C}_{B_0,B_1,T}$ is compact therefore $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}_{B_0,B_1,T}$ (as the unique accumulation point of the sequence). $\square$

Now we can apply Schauder's fixed point theorem to $\mathcal{F}$ and we get a solution

$$u \in L^\infty\big((0,T);\mathcal{C}^{1,1}([0,1])\big) \cap \mathrm{Lip}\big([0,T];H_0^1(0,1)\big).$$

The additional regularity properties of any solution u, meaning

$$\forall p > +\infty \quad u \in \mathcal{C}^0\big([0,T],W^{2,p}(0,1)\big) \cap C^1\big([0,1],W_0^{1,p}(0,1)\big),$$

follow directly from the construction of $\mathcal{F}$ and from Proposition A.8.

To obtain the minimum existence time announced we just have to realize that the only possible reduction of T occured in Section 2.3. This concludes the proof of Theorem 1.

2.5. Uniqueness

To conclude the part about the initial boundary value problem, we prove a weak-strong uniqueness property.

Theorem 4. *Let (y,u) and $(\tilde{y},\tilde{u})$ be two solutions of* (10) *and* (11) *for the same initial and boundary data, and such that $\tilde{y} \in L^\infty((0,T);\mathrm{Lip}([0,1]))$. Then $y = \tilde{y}$ and $u = \tilde{u}$.*

Proof. Define $Y = \tilde{y} - y$ and $U = \tilde{u} - u$. Then we have:

$$U \in \mathrm{Lip}\big([0,T];H_0^1(0,1)\big), \quad \big(1-\partial_{xx}^2\big)U(t,.) = Y(t,.)\ dt \text{ a.e.},$$

and $Y \in L^\infty(\Omega_T)$ is the unique weak solution of:

$$\partial_t Y + (u + \mathcal{A})\partial_x Y = -2.Y.\partial_x(u + \mathcal{A}) - \partial_x \tilde{y}.U - 2\tilde{y}.\partial_x U,$$

with $Y_0 = 0$, $Y_l = 0$, $Y_r = 0$. Using Theorem 6 and formula (50) we get with $b = -2.\partial_x(u + \mathcal{A})$ and $f = -U.\partial_x \tilde{y} - 2\tilde{y}.\partial_x U$:

$$\text{For } (t,x) \in P, \quad Y(t,x) = 0,$$

$$\text{For } (t,x) \in I, \quad Y(t,x) = \int_0^t f\big(r, \phi(r,t,x)\big).\exp\left(\int_r^t b\big(r', \phi\big(r',t,x\big)\big)\,dr'\right) dr,$$

$$\text{For } (t,x) \in L, \quad Y(t,x) = \int_{e(t,x)}^t f\big(r, \phi(r,t,x)\big).\exp\left(\int_r^t b\big(r', \phi\big(r',t,x\big)\big)\,dr'\right) dr,$$

$$\text{For } (t,x) \in R, \quad Y(t,x) = \int_{e(t,x)}^t f\big(r, \phi(r,t,x)\big).\exp\left(\int_r^t b\big(r', \phi\big(r',t,x\big)\big)\,dr'\right) dr.$$

Now since $\|U(t,.)\|_{L^\infty(0,1)} \leqslant 5.\|Y(t,.)\|_{L^\infty(0,1)}$ and $\tilde{y}$, $\partial_x \tilde{y}$ bounded, we see that for some $C > 0$:

$$\big\|f(t,.)\big\|_{L^\infty(0,1)} \leqslant C.\big\|Y(t,.)\big\|_{L^\infty(0,1)}\ dt \text{ a.e.,}$$

and since b is bounded, we get that for some $C' > 0$:

$$\big\|Y(t,.)\big\|_{L^\infty(0,1)} \leqslant C'.\int_0^t \big\|Y(s,.)\big\|_{L^\infty(0,1)}\,ds\ dt \text{ a.e.,}$$

and we conclude using Gronwall's lemma. □

3. Stabilization

In this part we prove Theorem 3. Here again we suppose that $\kappa \leqslant 0$. We begin by reformulating (2) and we also give the corresponding statement to Theorem 3 for this new formulation.

Rather than (2) we will work on:

$$\begin{cases} \partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2y.\partial_x(\check{u} + \check{\mathcal{A}}), \\ \big(1 - \partial_{xx}^2\big)\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0, \\ \big(1 - \partial_{xx}^2\big)\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = v_l(t) + \kappa, \quad \check{\mathcal{A}}(t,1) = v_r(t) + \kappa. \end{cases} \tag{29}$$

This system is equivalent to (2) with the change of unknown

$$v = \check{\mathcal{A}} + \check{u} - \kappa.$$

And our stationary feedback law still reads (15). One can check that Theorem 3 can be reformulated in terms of those new unknowns as:

Theorem 5. *Let $A_l > 2.\sinh(1)$, $A_r > A_l.\cosh(1) + \sinh(2)$, $M > 0$, $T > 0$. For any $y_0 \in \mathcal{C}^0([0,1])$ there exists $y \in \mathcal{C}^0(\Omega_T)$ such that if we define $\check{u}$ and $\check{\mathcal{A}}$ by*:

$$\forall (t,x) \in \Omega_T \quad \left(1 - \partial_{xx}^2\right)\check{u}(t,x) = y(t,x), \qquad \check{u}(t,0) = \check{u}(t,1) = 0,$$
$$\forall (t,x) \in \Omega_T \quad \left(1 - \partial_{xx}^2\right)\check{\mathcal{A}}(t,x) = 0, \qquad \check{\mathcal{A}}(t,0) = A_l.\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])} \quad \text{and}$$
$$\check{\mathcal{A}}(t,1) = A_r.\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])},$$

then y is the weak solution of

$$\partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2.y.\partial_x(\check{u} + \check{\mathcal{A}}). \tag{30}$$

This function y also satisfies:

$$\forall t \in [0,T] \quad \partial_t y(t,0) + M.y(t,0) = 0,$$
$$\forall x \in [0,1] \quad y(0,x) = y_0(x).$$

Besides, if y is a maximal solution of the closed loop system (15), (29) *then y is defined on $[0,+\infty) \times [0,1]$. And finally if we let $c = \min(A_l - 2.\sinh(1), \frac{A_r - A_l.\cosh(1) - \sinh(2)}{\sinh(1)})$ and $\tau = \frac{1}{M}.\ln(\frac{2.c.\|y_0\|_{\mathcal{C}^0([0,1])}}{M})$, we have*:

$$\forall t \geqslant \tau \quad \left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])} \leqslant \frac{M}{2c}.\frac{1}{1 + M(t-\tau)}. \tag{31}$$

We now prove Theorem 5.

3.1. Strategy

Let us first describe the main steps of the proof of Theorem 5. In terms of the new unknowns, the equilibrium state that we want to stabilize is $y = 0$, $\check{u} = \check{\mathcal{A}} = 0$. A first natural idea would be to look at the linearized system around the equilibrium state. Its stabilization would provide a local stabilization result on the nonlinear system. But the linearized system reads:

$$\begin{cases} \partial_t y - \kappa.\partial_x y = 0, \\ \left(1 - \partial_{xx}^2\right)\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0, \\ \left(1 - \partial_{xx}^2\right)\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = v_l(t) + \kappa, \quad \check{\mathcal{A}}(t,1) = v_r(t) + \kappa. \end{cases} \tag{32}$$

In the case $\kappa = 0$, the state y is constant therefore the system is not stabilizable.

In this situation we will apply a rough version of the return method that J.-M. Coron introduced in [10]. We will try to use the control in order to put the system in a simpler dynamic where it is easier to stabilize.

When we look at the transport equation we see that the sign of $\check{u} + \check{\mathcal{A}} - \kappa$ controls the geometry of the characteristics, and the sign of $\partial_x(\check{u} + \check{\mathcal{A}})$ controls the growth of y along the characteristics.

Therefore we would like our feedback law to provide $\breve{u} + \breve{\mathcal{A}} \geqslant 0$ (since $-\kappa \geqslant 0$) and $\partial_x(\breve{u} + \breve{\mathcal{A}}) \geqslant 0$. Considering the estimates ((33), (34)) on $\breve{u}$ we can get from the elliptic equation of (29) we see that with $v_l(t) = A_l.\|y(t,.)\|_{\mathcal{C}^0([0,1])} - \kappa$, $v_r(t) = A_r.\|y(t,.)\|_{\mathcal{C}^0([0,1])} - \kappa$, $\breve{\mathcal{A}}$ will dominate $\breve{u}$ and we will have the desired signs.

For the existence of a solution we cannot adapt our proof of existence for the initial boundary value problem completely. Our feedback law makes us lose some regularity in time because $\breve{\mathcal{A}}$ is now an unknown and it has exactly the time regularity of $\|y(t,.)\|_{\mathcal{C}^0([0,1])}$. To compensate for this, we will work in the space of continuous functions for y. This is now possible because the flow will always point toward $x = 1$. Therefore we have to prescribe y_l, and we just need to make a continuous transition at $(t,x) = (0,0)$ and have y_l decreasing in time. This is garanteed by $\partial_t y_l(t) + M.y_l(t) = 0$. In the next part we will prove the existence part of Theorem 5. The asymptotic properties will be proved in the last part.

3.2. Existence of a solution to the closed loop system

Once again, we use a fixed point strategy on an operator $\mathcal{S}$ we describe now. We begin by defining the domain of the operator.

Definition 3. Let X be the space of $(g,N) \in \mathcal{C}^0([0,T] \times [0,1]) \times \mathcal{C}^0([0,T])$ satisfying:

1. $\forall (t,x) \in [0,T] \times [0,1]$ $g(0,x) = y_0(x)$, $g(t,0) = y_0(0).e^{-M.t}$,
2. $\forall t \in [0,T]$ $\|g(t,.)\|_{\mathcal{C}^0([0,1])} \leqslant N(t)$,
3. N is nonincreasing and $N(0) \leqslant \|y_0\|_{\mathcal{C}^0([0,1])}$.

Proposition 3.1. *The domain X is nonempty, convex, bounded and closed with respect to the uniform topology.*

The proof is elementary and one notices that $(y_0(x).e^{-Mt}, \|y_0\|_{\mathcal{C}^0([0,1])}.e^{-Mt}) \in X$.

Now for $(y,N) \in X$ we define $\breve{u}$ and $\breve{\mathcal{A}}$ as the solutions of:

$$\forall (t,x) \in \Omega_T \quad \left(1 - \partial_{xx}^2\right)\breve{u}(t,x) = y(t,x) \quad \text{and} \quad \breve{u}(t,0) = \breve{u}(t,1) = 0,$$

$$\forall (t,x) \in \Omega_T \quad \left(1 - \partial_{xx}^2\right)\breve{\mathcal{A}}(t,x) = 0, \qquad \breve{\mathcal{A}}(t,0) = A_l N(t) \quad \text{and} \quad \breve{\mathcal{A}}(t,1) = A_r N(t).$$

One has the following exact formulas:

$$\forall (t,x) \in \Omega_T \quad \breve{u}(t,x) = -\int_0^x \sinh(x - \tilde{x}).y(t,\tilde{x})\,d\tilde{x},$$

$$\forall (t,x) \in \Omega_T \quad \breve{\mathcal{A}}(t,x) = \frac{N(t)}{\sinh(1)}.\big(A_r.\sinh(x) + A_l.\sinh(1-x)\big).$$

Therefore we have the following inequalities:

$$\forall (t,x) \in [0,T] \times [0,1] \quad \left|\breve{u}(t,x)\right| \leqslant 2\sinh(1)\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])}, \tag{33}$$

$$\left|\partial_x \breve{u}(t,x)\right| \leqslant 2\cosh(1)\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])},$$

$$\left|\partial^2_{xx}\check{u}(t,x)\right| \leqslant \bigl(1+2\sinh(1)\bigr)\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])}, \tag{34}$$

$$\left|\partial_x \check{\mathcal{A}}(t,x)\right| \geqslant \frac{A_r - 2\cosh(1)A_l}{\sinh(1)}.N(t), \qquad \left|\check{\mathcal{A}}(t,x)\right| \geqslant A_l.N(t). \tag{35}$$

And in turn those provide:

$$\forall (t,x)\in[0,T]\times[0,1] \quad (\check{u}+\check{\mathcal{A}})(t,x) \geqslant \bigl(A_l - 2.\sinh(1)\bigr).\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])}, \tag{36}$$

$$\forall (t,x)\in[0,T]\times[0,1]$$

$$\partial_x(\check{u}+\check{\mathcal{A}})(t,x) \geqslant \frac{A_r - 2.\cosh(1).A_l - \sinh(2)}{\sinh(1)}.\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])}. \tag{37}$$

Now if ϕ is the flow of $\check{u}+\check{\mathcal{A}}-\kappa$, ϕ is $\mathcal{C}^1$ and since $\check{u}+\check{\mathcal{A}}-\kappa \geqslant 0$ (thanks to the inequalities above), $\phi(.,t,x)$ is nondecreasing. This allows us to define the entrance time and then the operator $\mathcal{S}$ as follows. Let $e(t,x) = \min\{s\in[0,T] \mid \phi(s,t,x)=0\}$ with the convention that $\min\emptyset = 0$.

Now for $(t,x)\in[0,T]\times[0,1]$, $\mathcal{S}(y,N) = (\tilde{y},\tilde{N})$ with:

1. if $x \geqslant \phi(t,0,0)$ $\tilde{y}(t,x) = y_0(\phi(0,t,x)).\exp(-2\int_0^t \partial_x(\check{u}+\check{\mathcal{A}})(s,\phi(s,t,x))\,ds)$,
2. if $x \leqslant \phi(t,0,0)$ $\tilde{y}(t,x) = y_0(0).e^{-M.e(t,x)}.\exp(-2.\int_{e(t,x)}^t \partial_x(\check{u}+\check{\mathcal{A}})(s,\phi(s,t,x))\,ds)$,
3. $\tilde{N}(t) = \|\tilde{y}(t,.)\|_{\mathcal{C}^0([0,1])}$.

From Theorem 6 we know that $\tilde{y}$ is the weak solution of:

$$\partial_t\tilde{y} + (\check{u}+\check{\mathcal{A}}-\kappa)\partial_x\tilde{y} = -2\tilde{y}\partial_x(\check{u}+\check{\mathcal{A}}), \qquad \tilde{y}(0,.) = y_0, \qquad \tilde{y}(t,0) = y_0(0)e^{-Mt}. \tag{38}$$

Before applying Schauder's fixed point theorem to $\mathcal{S}$ we prove the following statements.

Proposition 3.2.

1. *The operator $\mathcal{S}$ maps X to X.*
2. *The family $\mathcal{S}(X)$ is uniformly bounded and equicontinuous.*
3. *$\mathcal{S}$ is continuous w.r.t. the uniform topology.*

Proof.

1. It will be useful to distinguish the cases where $y_0(0)=0$ (case 1) and $y_0(0)\neq 0$ (case 2). First remark that $\tilde{y}$ being continuous, $\tilde{N}$ is continuous. Now in case 1, we have that $\forall (t,x)\in\Omega_T$, $x\leqslant\phi(t,0,0) \Rightarrow \tilde{y}(t,x)=0$ and both the continuity on $\{(t,x)\in\Omega_T \mid x>\phi(t,0,0)\}$ and the continuity at the interface $\{(t,x)\in\Omega_T \mid x=\phi(t,0,0)\}$ are obvious.
   In case 2, one must first remark that $\forall t\in[0,T]$, $y(t,0)\neq 0$, so $\forall t\in[0,T]$, $0 < \|y(t,.)\|_{\mathcal{C}^0([0,1])} \leqslant N(t)$. This implies that every characteristic curve points to the right and so e corresponds to Definition A.1. Therefore e is $\mathcal{C}^1$ on $\{(t,x)\in\Omega_T \mid x<\phi(t,0,0)\}$ and continuous at the interface $\{(t,x)\in\Omega_T \mid x=\phi(t,0,0)\}$, once again we see that $\tilde{y}$ is continuous in Ω_T, and so is $\tilde{N}$.
   Now it is straightforward from its definition that

$$\forall (t,x)\in[0,T]\times[0,1], \quad \tilde{y}(0,x) = y_0(x), \qquad \tilde{y}(t,0) = y_0(0).e^{-M.t}.$$

It only remains to see that $\tilde{N} = \|\tilde{y}(t,.)\|_{\mathcal{C}^0([0,1])}$ is nonincreasing. Since $\partial_x(\check{u} + \check{\mathcal{A}}) \geqslant 0$ (see (37)), we see from the definition of $\tilde{y}$ that $|\tilde{y}|$ does not increase along the characteristics, and since $|\tilde{y}(t,0)|$ is also nonincreasing we can conclude.

2. Since X is already bounded and thanks to the first part of the proof, $\mathcal{S}(X)$ is bounded. The equicontinuity of the family $\{\tilde{N}\}$ being implied by the one of the family $\{\tilde{y}\}$, we will show that we have a common continuity modulus for all $\{\tilde{y}\}$. For now let us focus only on $\{(t,x) \in \Omega_T \mid x \leqslant \phi(t,0,0)\}$. On this set $\tilde{y}(t,x) = 0$ in case 1. In the second case, we need the following inequalities valid on Ω_T and which follow from the definition of $\check{u}$ and $\check{\mathcal{A}}$:

$$\|\check{u}\|_{\mathcal{C}^0(\Omega_T)} \leqslant 2.\sinh(1).\|y_0\|_{\mathcal{C}^0([0,1])}, \tag{39}$$

$$\|\partial_x\check{u}\|_{\mathcal{C}^0(\Omega_T)} \leqslant 2.\cosh(1).\|y_0\|_{\mathcal{C}^0([0,1])}, \tag{40}$$

$$\left\|\partial^2_{xx}\check{u}\right\|_{\mathcal{C}^0(\Omega_T)} \leqslant \big(1 + 2.\sinh(1)\big).\|y_0\|_{\mathcal{C}^0([0,1])}, \tag{41}$$

$$\|\check{\mathcal{A}}\|_{\mathcal{C}^0(\Omega_T)} = \left\|\partial^2_{xx}\check{\mathcal{A}}\right\|_{\mathcal{C}^0(\Omega_T)} \leqslant (A_r + A_l)\|y_0\|_{\mathcal{C}^0([0,1])}, \tag{42}$$

$$\|\partial_x\check{\mathcal{A}}\|_{\mathcal{C}^0(\Omega_T)} \leqslant \frac{A_r + A_l}{\tanh(1)}.\|y_0\|_{\mathcal{C}^0([0,1])}. \tag{43}$$

And since ϕ is the flow of $\check{u} + \check{\mathcal{A}} - \kappa$ we also have:

$$\begin{aligned}
\|\partial_1\phi\|_{\mathcal{C}^0([0,1])} &\leqslant -\kappa + \big(2\sinh(1) + A_l + A_r\big)\|y_0\|_{\mathcal{C}^0([0,1])},\\
\|\partial_2\phi\|_{\mathcal{C}^0([0,1])} &\leqslant \big(-\kappa + \big(2\sinh(1) + A_l + A_r\big)\|y_0\|_{\mathcal{C}^0([0,1])}\big)\\
&\quad\times \exp\left(2.T.\cosh(1).\left(2 + \frac{A_r + A_l}{\sinh(1)}\right)\|y_0\|_{\mathcal{C}^0([0,1])}\right),\\
\|\partial_3\phi\|_{\mathcal{C}^0([0,1])} &\leqslant \exp\left(2.T.\cosh(1).\left(2 + \frac{A_r + A_l}{\sinh(1)}\right)\|y_0\|_{\mathcal{C}^0([0,1])}\right).
\end{aligned}$$

Now since we have

$$\tilde{y}(t,x) = y_0(0).e^{-M.e(t,x)}.\exp\left(-2.\int_{e(t,x)}^{t} \partial_x(\check{u} + \check{\mathcal{A}})\big(r,\phi(r,t,x)\big)\,dr\right),$$

we see that we only need a uniform bound on $\|e\|_{\mathcal{C}^1}$ to conclude about the equicontinuity on $\{(t,x) \in \Omega_T \mid x \leqslant \phi(t,0,0)\}$.

We have $0 \leqslant e(t,x) \leqslant T$, and thanks to the definition of e, to (39), (42) and $\|y(t,.)\|_{\mathcal{C}^0([0,1])} \geqslant |y(t,0)| = |y_0(0)|.e^{-M.t} \geqslant |y_0(0).e^{-M.T}|$ we get:

$$\begin{aligned}
&\big|\partial_t e(t,x)\big|\\
&\quad\leqslant \frac{(\kappa + (2\sinh(1) + A_l + A_r)\|y_0\|_{\mathcal{C}^0([0,1])}).\exp(2.T.\cosh(1).(2 + \frac{A_r+A_l}{\sinh(1)})\|y_0\|_{\mathcal{C}^0([0,1])})}{(A_l - 2\sinh(1)).e^{-M.T}.|y_0(0)|}.
\end{aligned}$$

In the same way:

$$\left|\partial_x e(t,x)\right| \leqslant \frac{\exp(2.T.\cosh(1).(2+\frac{A_r+A_l}{\sinh(1)})\|y_0\|_{\mathcal{C}^0([0,1])})}{(A_l-2\sinh(1)).e^{-M.T}.|y_0(0)|}.$$

In the end, we see that both in case 1 and case 2, the family $\{\tilde{y}\}$ is uniformly Lipschitz on $\{(t,x)\in\Omega_T \mid x\leqslant\phi(t,0,0)\}$. Now on $\{(t,x)\in\Omega_T \mid x\geqslant\phi(t,0,0)\}$, we know

$$\tilde{y}(t,x)=y_0\big(\phi(0,t,x)\big).\exp\left(-2.\int_0^t \partial_x(\check{u}+\check{\mathcal{A}})\big(r,\phi(r,t,x)\big)\,dr\right).$$

Clearly y_0 is continuous on $[0,1]$ therefore it is both bounded and uniformly continuous, the family of functions ϕ is uniformly Lipschitz and the family $\{\exp(-2.\int_0^t \partial_x(\check{u}+\check{\mathcal{A}})\times(r,\phi(r,t,x))\,dr)\}$ is uniformly bounded and equicontinuous. We can conclude that the family $\{\tilde{y}\}$ is also equicontinuous on $\{(t,x)\in\Omega_T \mid x\geqslant\phi(t,0,0)\}$. Since we have continuity on $\{(t,x)\in\Omega_T \mid x=\phi(t,0,0)\}$, we can conclude that the family $\mathcal{S}(X)$ is uniformly bounded and equicontinuous on Ω_T, $\mathcal{S}(X)$ is therefore relatively compact in X.

3. It remains to prove that $\mathcal{S}$ is continuous w.r.t. to the uniform convergence.
Let (y_n) be a sequence in X converging uniformly to $y\in X$. We only have to show that $\tilde{y}_n$ converges uniformly to $\tilde{y}$, since it immediately implies that $\tilde{N}_n$ converges uniformly to $\tilde{N}$. First the uniform convergence of y_n and N_n implies the uniform convergence of $\check{u}_n$ and $\check{\mathcal{A}}_n$. Then by Gronwall's lemma, we also have $\phi_n\to\phi$ uniformly in $\mathcal{C}^1(\Omega_T)$. Using Proposition A.2, we then obtain $e_n\to e$ uniformly in $\mathcal{C}^0(\Omega_T)$. Now we decompose Ω_T in three parts depending on n.

$$L_n=\left\{(t,x)\in\Omega_T \;\middle|\; x\leqslant\min\big(\phi_n(t,0,0),\phi(t,0,0)\big)\right\},$$
$$R_n=\left\{(t,x)\in\Omega_T \;\middle|\; x\geqslant\max\big(\phi_n(t,0,0),\phi(t,0,0)\big)\right\},$$
$$I_n=\overline{\Omega_T\setminus(L_n\cup R_n)}.$$

Let us point out first that when $n\to+\infty$:

$$\liminf L_n=\left\{(t,x)\in\Omega_T \;\middle|\; x\leqslant\phi(t,0,0)\right\},\qquad \liminf R_n=\left\{(t,x)\in\Omega_T \;\middle|\; x\geqslant\phi(t,0,0)\right\},$$
$$\text{and}\quad \limsup I_n=\left\{(t,x)\in\Omega_T \;\middle|\; x=\phi(t,0,0)\right\}.$$

- For $(t,x)\in L_n$ if $y_0(0)=0$ then y_n and $\tilde{y}$ are equal to zero otherwise we have the formulas:

$$\tilde{y}(t,x)=y_0(0).e^{-M.e(t,x)}.\exp\left(-2\int_{e(t,x)}^t \partial_x(\check{u}+\check{\mathcal{A}})\big(r,\phi(r,t,x)\big)\,dr\right),$$

$$\tilde{y}_n(t,x)=y_0(0).e^{-M.e_n(t,x)}.\exp\left(-2\int_{e_n(t,x)}^t \partial_x(\check{u}_n+\check{\mathcal{A}}_n)\big(r,\phi_n(r,t,x)\big)\,dr\right),$$

and the uniform convergence of $\tilde{y}_n$ follows from the uniform boundedness and convergence of $\partial_x \check{u}_n$, $\partial_x \check{\mathcal{A}}_n$, e_n and ϕ_n.
- For $(t,x) \in R_n$ the proof is similar.
- It remains only to prove the convergence in I_n. But the width of I_n tends to zero, and the family $\{\tilde{y}_n\}$ is equicontinuous. Therefore the uniform convergence of $\tilde{y}_n$ in I_n follows from those in L_n and R_n. □

Now we can apply Schauder's fixed point theorem to $\mathcal{S}$ and get (y,N) fixed point of $\mathcal{S}$. It remains to show that it satisfies all of the properties of Theorem 5 except (31) which will be proven in the next subsection. First we have $y(t,0) = \tilde{y}(t,0) = y_0(0).e^{-M.t}$ and it implies $\partial_t y(t,0) = -M.y(t,0)$.

But also $N(t) = \tilde{N}(t) = \|\tilde{y}(t,.)\|_{\mathcal{C}^0([0,1])} = \|y(t,.)\|_{\mathcal{C}^0([0,1])}$, therefore $\|y(t,.)\|_{\mathcal{C}^0([0,1])}$ is nonincreasing and, thanks to Theorem 6, $y = \tilde{y}$ is a weak solution of

$$\begin{cases} (1-\partial_{xx}^2)\check{u} = y, \quad \check{u}(t,0) = \check{u}(t,1) = 0, \\ (1-\partial_{xx}^2)\check{\mathcal{A}} = 0, \quad \check{\mathcal{A}}(t,0) = A_l.\|y(t,.)\|_{\mathcal{C}^0([0,1])}, \quad \check{\mathcal{A}}(t,1) = A_r.\|y(t,.)\|_{\mathcal{C}^0([0,1])}, \\ \partial_t y + (\check{u} + \check{\mathcal{A}} - \kappa).\partial_x y = -2y.\partial_x(\check{u} + \check{\mathcal{A}}). \end{cases} \tag{44}$$

Remark 5.

- Since $(\check{u} + \check{\mathcal{A}} - \kappa)(t,1) = A_r.\|y(t,.)\|_{\mathcal{C}^0([0,1])} - \kappa \geqslant 0$ we had all along $\Gamma_r = \emptyset$.
- Since $(\check{u} + \check{\mathcal{A}} - \kappa)(t,0) = A_l.\|y(t,.)\|_{\mathcal{C}^0([0,1])} - \kappa$, we see that a priori, Γ_l depends on y. But in fact if $y_0(0) \neq 0$ then $\forall t$, $y(t,0) \neq 0$ and $\Gamma_l = \mathbb{R}+$. And if $y_0(0) = 0$ then $\forall t$, $y_l(t) = y(t,0) = 0$ and it makes no difference in the weak formulation (53) if we enlarge Γ_l to $\mathbb{R}+$. Therefore the space of test functions is always:

$$Adm(\Omega_T) = \{\phi \in \mathcal{C}^1(\Omega_T) \mid \forall x \in [0,1]\ \phi(T,x) = 0,\ \forall t \in [0,T]\ \phi(t,1) = 0\}.$$

- It must be noted that while we required $T < \infty$, we did not need T to be small.

3.3. Stabilization and global existence

To finish the proof of Theorem 5 we have to prove the global existence of a maximal solution and estimate (31).

Proof. First we rewrite (36), (37) as:

$$\forall (t,x) \in \Omega_T \qquad (\check{u} + \check{\mathcal{A}})(t,x) \geqslant c\|y(t,.)\|_{\mathcal{C}^0([0,1])},$$
$$\partial_x\big((\check{u} + \check{\mathcal{A}})(t,x)\big) \geqslant c\|y(t,.)\|_{\mathcal{C}^0([0,1])}.$$

But y is the solution of the transport equation (30) and it satisfies:

$$y(t,x) = y\big(s,\phi(s,t,x)\big).\exp\left(-2\int_s^t \partial_x(\check{u} + \check{\mathcal{A}})\big(r,\phi(r,t,x)\big)\,dr\right).$$

Combining those facts, we get for $t \geqslant s$:

$$\left|y(t,x)\right| \leqslant \left|y\big(s,\phi(s,t,x)\big)\right|.\exp\left(-2\int_s^t c.\left\|y(r,.)\right\|_{\mathcal{C}^0([0,1])}\,dr\right).$$

This implies that $|y|$ decreases along the characteristics (strictly for the times where $y(t,.) \not\equiv 0$). But we have also imposed $y(t,0) = y(s,0).e^{-M(t-s)}$, therefore $|y|$ also decreases along $x = 0$. This already shows, thanks to the existence theorem that a maximal solution of the closed loop system is global. To get a more precise statement, we consider all the characteristics between time t and s and we obtain:

$$\text{for } 0 \leqslant s \leqslant t \quad \left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])} \leqslant \left\|y(s,.)\right\|_{\mathcal{C}^0([0,1])}.\max_{r\in[s,t]}\left(e^{-M(r-s)}.\exp\left(-2c\int_r^t \left\|y(\alpha,.)\right\|_{\mathcal{C}^0([0,1])}\,d\alpha\right)\right).$$

Now we define $g(r) = e^{-M(r-s)}.\exp(-2c\int_r^t \|y(\alpha,.)\|_{\mathcal{C}^0([0,1])}\,d\alpha)$, then $g'(r) = (2c\|y(r,.)\|_{\mathcal{C}^0([0,1])} - M)g(r)$ and we know that as long as the quantity $\|y(r,.)\|_{\mathcal{C}^0([0,1])}$ is not equal to zero, it strictly decreases. So if $\|y_0\|_{\mathcal{C}^0([0,1])} > \frac{M}{2c}$, for t small enough $\|y(t,.)\|_{\mathcal{C}^0([0,1])} \geqslant \frac{M}{2c}$ and we have:

$$\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])} \leqslant \|y_0\|_{\mathcal{C}^0([0,1])}.e^{-M.t}$$

which implies $\|y(\tau,.)\|_{\mathcal{C}^0([0,1])} \leqslant \frac{M}{2c}$. This provides for $\tau \leqslant s \leqslant t$, the inequality (which was clear when $\|y_0\|_{\mathcal{C}^0([0,1])} \leqslant \frac{M}{2c}$)

$$\left\|y(t,.)\right\|_{\mathcal{C}^0([0,1])} \leqslant \left\|y(s,.)\right\|_{\mathcal{C}^0([0,1])}.\exp\left(-2c\int_s^t \left\|y(r,.)\right\|_{\mathcal{C}^0([0,1])}\,dr\right).$$

And we conclude with a classical comparison principle for ODES. □

Remark 6.

- For $\kappa \neq 0$ the result is easily improved.
  Indeed if $t \geqslant \tau - \frac{2\sinh(1)+A_l+A_r}{\kappa.c}$ we have $-\kappa + \check{u} + \check{\mathcal{A}} \geqslant -\frac{\kappa}{2}$.
  And therefore $t \geqslant \tau - \frac{2\sinh(1)+A_l+A_r}{\kappa.c} - \frac{2}{\kappa} \Rightarrow \|y(t,.)\|_{\mathcal{C}^0([0,1])} \leqslant |y_0(0)|.e^{-M(t+\frac{2}{\kappa})}$.
- In particular if $y_0(0) = 0$ we see that we stabilize the null state in finite time.
- Of course similar results hold for $\kappa \geqslant 0$ thanks to Remark 2.

Appendix A. Initial boundary value problem for a linear transport equation

In this section we will consider the initial boundary value problem for the following linear transport equation:

$$\partial_t y + a(t,x).\partial_x y = b(t,x).y + f(t,x). \tag{45}$$

We will look at strong and weak solutions of (45) on $\Omega_T = [0,T]\times[0,1]$. It should be noted that the backward problem is transformed in a standard one by the change of variables: $t \to T-t$.

A.1. Properties of the flow

Let $a \in \mathcal{C}^0(\Omega_T)$ be uniformly Lipschitz in the second variable with constant $L = \|a\|_{L^\infty((0,T),\mathrm{Lip}([0,1]))}$. Since we want to use the method of characteristics to solve (45) we need to study the flow of a.

Definition 4. For $(t,x)\in\Omega_T$, let $\phi(.,t,x)$ be the $\mathcal{C}^1$ maximal solution to:

$$\begin{cases} \partial_s\phi(s,t,x) = a\big(s,\phi(s,t,x)\big), \\ \phi(t,t,x) = x, \end{cases} \tag{46}$$

which is defined on a certain set $[e(t,x),h(t,x)]$ (which is closed because $[0,1]$ is compact) and with possibly $e(t,x)$ and/or $h(t,x)=t$.

Remark 7. Obviously $e(t,x)>0 \Rightarrow \phi(e(t,x),t,x)\in\{0,1\}$.

Now we take into account the influence of the boundaries by introducing the sets:

$$\begin{aligned} P = \big\{(t,x)\in\Omega_T \,\big|\, \exists s\in\big[e(t,x),h(t,x)\big] \text{ such that } \phi(s,t,x)\in\{0,1\} \text{ and } a\big(s,\phi(s,t,x)\big)=0\big\} \\ \cup\big\{\big(s,\phi(s,0,0)\big)\,\big|\,\forall s\in[0,T]\big\}\cup\big\{\big(s,\phi(s,0,1)\big)\,\big|\,\forall s\in[0,T]\big\}, \end{aligned}$$

$$I = \big\{(t,x)\in\Omega_T\setminus P \,\big|\, e(t,x)=0\big\},$$

$$L = \big\{(t,x)\in\Omega_T\setminus P \,\big|\, \phi\big(e(t,x),t,x\big)=0\big\},$$

$$R = \big\{(t,x)\in\Omega_T\setminus P \,\big|\, \phi\big(e(t,x),t,x\big)=1\big\},$$

$$\Gamma_l = \big\{t\in[0,T]\,\big|\, a(t,0)>0\big\},$$

$$\Gamma_r = \big\{t\in[0,T]\,\big|\, a(t,1)<0\big\}.$$

Proposition A.1. *The function ϕ is uniformly Lipschitz on its domain.*

Proof. This is easily deduced from the standard case by the use of a Lipschitzian extension of a. □

We can now study the regularity of e.

Proposition A.2. *Let $(t,x)\in\Omega_T\setminus P$, $(a_n)\in\mathcal{C}^0(\Omega_T)\cap L^\infty((0,T);\mathrm{Lip}([0,1]))$ a sequence such that $\|a_n-a\|_{\mathcal{C}^0(\Omega_T)}\to 0$, $\|a_n\|_{L^\infty(0,1;\mathrm{Lip}([0,1]))}$ is bounded and $(t_n;x_n)\in\Omega_T$ such that $(t_n,x_n)\to(t,x)$. Then $e_n(t_n,x_n)\to e(t,x)$.*

Proof. Once again we will use a Lipschitzian extension operator Π and we set $\tilde{a}_n = \Pi(a_n)$ and $\tilde{a} = \Pi(a)$. Now let $\tilde{\phi}_n$ and $\tilde{\phi}$ be their respective flows. Using Gronwall's lemma we have:

$$\left|(\tilde{\phi}_n - \tilde{\phi})(s,t,x)\right| \leqslant T.\|\tilde{a}_n - \tilde{a}\|_{\mathcal{C}^0(\Omega_T)}.e^{T\cdot\|\tilde{a}\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}}. \tag{47}$$

But we can see that:

$$e_n(t_n,x_n) = \min\left\{s \in [0,t_n] \,\middle|\, \forall r \in [s,t_n],\ \tilde{\phi}_n(r,t,x) \in [0,1]\right\}.$$

- If $(t,x) \in I$ since we have excluded the characteristics coming from $(0,0)$ and $(0,1)$ we have that $\inf_{s\in[0,T]}(\mathrm{d}(\phi(s,t,x),[0,t]\times\{0\}\cup[0,t]\times\{1\})) > 0$. So we can conclude from (47) that for n large enough $\phi_n(.,t,x)$ is defined back to 0 that is $e_n(t,x) = 0$. From now on $(t,x) \in L \cup R$.
- Now we can take s strictly lower and close enough to $e(t,x)$, $\tilde{\phi}(s,t,x) \notin [0,1]$, since $(t,x) \notin P \Rightarrow e(t,x) \in \Gamma_l \cup \Gamma_r$. But $\tilde{\phi}_n(s,t_n,x_n) \to \tilde{\phi}(s,t,x)$, therefore for n large enough $\tilde{\phi}_n(s,t_n,x_n) \notin [0,1]$ and $s < t_n$ and we can conclude that $\liminf e_n(t_n,x_n) \geqslant s$. But s is arbitrarily close to $e(t,x)$ and we get
$$\liminf e_n(t_n,x_n) \geqslant e(t,x).$$
- If $e(t,x) = t$ then $\limsup e_n(t_n,x_n) \leqslant \limsup t_n = t$ and $e_n(t_n,x_n) \to e(t,x)$. Otherwise since $(t,x) \notin P$ then $\forall s \in\,]e(t,x),t[$, $\phi(s,t,x) \in\,]0,1[$. And now $\forall \epsilon > 0$, $\exists \alpha > 0$ such that $\forall s \in [e(t,x)+\epsilon, t-\epsilon] \min(\phi(s,t,x), 1-\phi(s,t,x)) \geqslant \alpha$. But for n large enough we have:
$$\|\phi_n - \phi\|_{\mathcal{C}^0(\Omega_T)} \leqslant \frac{\alpha}{4},$$
$$\left|\phi_n(s,t_n,x_n) - \phi_n(s,t,x)\right| \leqslant \frac{\alpha}{4}$$
(the second estimate comes from the uniform bound on $\|a_n\|_{L^\infty((0,1);\mathrm{Lip}([0,1]))}$). But now, combining those two inequalities we see that for n large and for all s between $e(t,x)+\epsilon$ and $t-\epsilon$ we have $\min(\phi_n(s,t_n,x_n), 1-\phi_n(s,t_n,x_n)) \geqslant \frac{\alpha}{2}$, this provides $\limsup e_n(t_n,x_n) \leqslant e(t,x)+\epsilon$, and since ϵ is arbitrarily small we obtain:
$$\limsup e_n(t_n,x_n) \leqslant e(t,x). \qquad \square$$

Remark 8.

- For $a_n = a$ it shows that e is continuous outside of P.
- If $P = \emptyset$, since Ω_T is compact the proposition implies that e_n converges uniformly toward e.

Proposition A.3. *If we assume that $\partial_x a \in \mathcal{C}^0(\Omega_T)$ then ϕ is $\mathcal{C}^1$ and e is $\mathcal{C}^1$ on $\Omega_T \setminus P$ with*:

$$\partial_t e(t,x) = \frac{a(t,x).\exp(\int_{e(t,x)}^s \partial_x a(r,\phi(r,t,x))dr)}{a(e(t,x),\phi(e(t,x),t,x))},$$
$$\partial_x e(t,x) = -\frac{\exp(\int_{e(t,x)}^s \partial_x a(r,\phi(r,t,x))dr)}{a(e(t,x),\phi(e(t,x),t,x))}. \tag{48}$$

Proof. The regularity of ϕ is a classical result. If $(t,x) \in I$, $e(t,x) = 0$ and it is obvious. For $(t,x) \in L$ we have $\phi(e(t,x),t,x) = 0$ and $e(t,x) \in \Gamma_l$ therefore $\partial_1\phi(e(t,x),t,x) > 0$ and the implicit function theorem let us conclude, we can proceed in the same way for R. The inclusion of the characteristics of $(0,0)$ and $(0,1)$ in P is needed here. □

Proposition A.4. *Let (a_n) be a sequence of $\mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$ and $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$ such that $\|a_n - a\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))} \underset{n\to+\infty}{\longrightarrow} 0$. If we call ϕ_n the flow of a_n and ϕ the flow of a then $\phi_n \underset{n\to+\infty}{\longrightarrow} \phi$ locally in $\mathcal{C}^1$.*

Proof. Once again using a $\mathcal{C}^1$ extension operator on a_n and a we deduce the result from the classical standard case, which follows from applications of Gronwall's lemma. □

A.2. Strong solutions

Here we consider the case of data $a \in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $y_l \in \mathcal{C}^1_c(\Gamma_l)$, $y_r \in \mathcal{C}^1_c(\Gamma_r)$, $y_0 \in \mathcal{C}^1_c(0,1)$, $b \in \mathcal{C}^1(\Omega_T)$ and $f \in \mathcal{C}^1_c(\Omega_T \setminus P)$. We define the function y in the following way:

$$\text{for } (t,x) \in P \quad y(t,x) = 0, \tag{49}$$

$$\text{for } (t,x) \in I \quad y(t,x) = y_0\big(\phi(0,t,x)\big).\exp\left(\int_0^t b\big(r,\phi(r,t,x)\big)\,dr\right) + \int_0^t f\big(r,\phi(r,t,x)\big).\exp\left(\int_r^t b\big(r',\phi(r',t,x)\big)\,dr'\right)dr,$$

$$\text{for } (t,x) \in L \quad y(t,x) = y_l\big(e(t,x)\big).\exp\left(\int_{e(t,x)}^t b\big(r,\phi(r,t,x)\big)\,dr\right) + \int_{e(t,x)}^t f\big(r,\phi(r,t,x)\big).\exp\left(\int_r^t b\big(r',\phi(r',t,x)\big)\,dr'\right)dr, \tag{50}$$

$$\text{for } (t,x) \in R \quad y(t,x) = y_r\big(e(t,x)\big).\exp\left(\int_{e(t,x)}^t b\big(r,\phi(r,t,x)\big)\,dr\right) + \int_{e(t,x)}^t f\big(r,\phi(r,t,x)\big).\exp\left(\int_r^t b\big(r',\phi(r',t,x)\big)\,dr'\right)dr.$$

Proposition A.5. *We have $y \in \mathcal{C}^1(\Omega_T)$, $\mathrm{supp}(y) \subset \Omega_T \setminus P$ and y is a strong solution of (45) with the additional conditions that for all x in $[0,1]$ $y(0,x) = y_0(x)$, for all t in Γ_l $y(t,0) = y_l(t)$ and for all t in Γ_r $y(t,1) = y_r(t)$. Besides we have the estimate*:

$$\|y\|_{\mathcal{C}^0(\Omega_T)} \leqslant \big(\max(\|y_0\|_{\mathcal{C}^0(0,1)}, \|y_l\|_{\mathcal{C}^0(\Gamma_l)}, \|y_r\|_{\mathcal{C}^0(\Gamma_r)}) + T.\|f\|_{\mathcal{C}^0(\Omega_T)}\big).e^{T.\|b\|_{\mathcal{C}^0(\Omega_T)}}. \tag{51}$$

Proof. First, y is equal to 0 in a neighbourhood of P because we chose y_0, y_l, y_r, f to be null close to P and because of (50). Outside of this neighbourhood, the regularity of y comes from the integral formulas (50) and from the regularity of y_0, y_l, y_r, f, b, ϕ and e (proved in Proposition A.3). The fact that y satisfies (45) is a straightforward calculation. □

Remark 9. We have that:

$$\begin{aligned}&\forall (t,x)\in \Omega_T \text{ and } \forall s\in \big[e(t,x),h(t,x)\big]\\ &\quad y(t,x)=y\big(s,\phi(s,t,x)\big).\exp\left(\int_s^t b\big(r,\phi(r,t,x)\big)\,dr\right)\\ &\qquad\qquad +\int_s^t f\big(r,\phi(r,t,x)\big).\exp\left(\int_r^t b\big(r',\phi(r',t,x)\big)\,dr'\right)dr. \end{aligned}\tag{52}$$

A.3. Weak solutions

In this section we will consider the case of data $a\in \mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $b, f\in L^\infty(\Omega_T)$, $y_0\in L^\infty(0,1)$, $y_l\in L^\infty(\Gamma_l)$ and $y_r\in L^\infty(\Gamma_r)$. We introduce the space of test functions:

$$\begin{aligned}Adm(\Omega_T)=\big\{\phi\in \mathcal{C}^1(\Omega_T)\;\big|\;&\forall x\in[0,1]\;\phi(T,x)=0, \forall t\in[0,T]\setminus\Gamma_l\;\phi(t,0)=0,\\ &\forall t\in[0,T]\setminus\Gamma_r\;\phi(t,1)=0\big\}.\end{aligned}$$

Proposition A.6. *For $y\in\mathcal{C}^1(\Omega_T)$, y is a strong solution of* (45)*, if and only if it satisfies $\forall\phi\in Adm(\Omega_T)$*

$$\begin{aligned}&\int_{\Omega_T} y.\big(\partial_t\phi+a.\partial_x\phi+(b+\partial_x a)\phi\big)\,dx\,dt\\ &\quad=-\int_{\Omega_T} f(t,x).\phi(t,x)\,dt\,dx-\int_0^1\phi(0,x).y(0,x)\,dx\\ &\qquad+\int_0^T\big(a(t,1).\phi(t,1).y(t,1)-a(t,0).\phi(t,0).y(t,0)\big)\,dt.\end{aligned}\tag{53}$$

This legitimates the following definition of a weak solution.

Definition 5. For $a\in L^\infty(0,T,\mathrm{Lip}(0,1))$, $b, f\in L^1(\Omega_T)$, $y_0\in L^1(0,1)$, $y_l\in L^1(\Gamma_l)$ and $y_r\in L^1(\Gamma_r)$, we say that $y\in L^\infty(\Omega_T)$ is a weak solution of (45) if it satisfies (53).

Theorem 6. *Let $a\in\mathcal{C}^0([0,T];\mathcal{C}^1([0,1]))$, $b, f\in L^\infty(\Omega_T)$, $y_0\in L^\infty(0,1)$, $y_l\in L^\infty(\Gamma_l)$ and $y_r\in L^\infty(\Gamma_r)$. We will also suppose that the sets*

$$P_l=\big\{t\in[0,T]\;\big|\;a(t,0)=0\big\}\quad\textit{and}\quad P_r=\big\{t\in[0,T]\;\big|\;a(t,1)=0\big\}$$

have at most a countable number of connected components. Then the function y defined by the formula (50)*, is a weak solution of* (45) *and satisfies*:

$$\|y\|_{L^\infty(\Omega_T)} \leqslant \big(\max\big(\|y_0\|_{L^\infty(0,1)}, \|y_l\|_{L^\infty(\Gamma_l)}, \|y_r\|_{L^\infty(\Gamma_r)}\big) + T.\|f\|_{L^\infty(\Omega_T)}\big).e^{T.\|b\|_{L^\infty(\Omega_T)}}. \tag{54}$$

Proof. If we let $P_{\tilde{t}} = P \cap \{(t,x) \in \Omega_T \mid t = \tilde{t}\}$, we can see that each points of a $P_{\tilde{t}}$ corresponds to at least one connected component of $P_l \cup P_r$ (since only one characteristic curve goes through the whole connected component) therefore, $P_{\tilde{t}}$ is at most countable and thus 1d negligible, this implies that P is 2d negligible.

Now we have:

- $\mathcal{C}_c^1(\Omega_T \setminus P)$ is dense in $L^1(\Omega_T)$,
- $\mathcal{C}_c^1(0,1)$ is dense in $L^1(0,1)$,
- $\mathcal{C}_c^1(\Gamma_l)$ is dense in $L^1(\Gamma_l)$,
- $\mathcal{C}_c^1(\Gamma_r)$ is dense in $L^1(\Gamma_r)$.

And we can take, thanks to the hypothesis on b, f, y_0, y_l and y_r:

- $(b_n) \in \mathcal{C}^1(\Omega_T)$ such that $\|b_n - b\|_{L^1(\Omega_T)} \to 0$ and $\|b_n\|_{L^\infty(\Omega_T)}$ is bounded,
- $(f_n) \in \mathcal{C}_c^1(\Omega_T \setminus P)$ such that $\|f_n - f\|_{L^1(\Omega_T)} \to 0$ and $\|f_n\|_{L^\infty(\Omega_T)}$ is bounded,
- $(y_{0,n}) \in \mathcal{C}_c^1(0,1)$ such that $\|y_{0,n} - y_0\|_{L^1(0,1)} \to 0$ and $\|y_{0,n}\|_{L^\infty(0,1)}$ is bounded,
- $(y_{l,n}) \in \mathcal{C}_c^1(\Gamma_l)$ such that $\|y_{l,n} - y_l\|_{L^1(\Gamma_l)} \to 0$ and $\|y_{l,n}\|_{L^\infty(\Gamma_l)}$ is bounded,
- $(y_{r,n}) \in \mathcal{C}_c^1(\Gamma_r)$ such that $\|y_{r,n} - y_l\|_{L^1(\Gamma_r)} \to 0$ and $\|y_{r,n}\|_{L^\infty(\Gamma_r)}$ is bounded.

We call (y_n) the sequence of strong solutions to (45). Thanks to (51) we can extract so that:

$$\exists y \in L^\infty(\Omega_T) \text{ such that } y_n \text{ converges to } y \text{ for the weak-}{*}\text{ topology of } L^\infty(\Omega_T).$$

Now we take the limit in (53) and conclude that y is a weak solution to (45).

We can also suppose (we just need to extract again) that we have pointwise convergence almost everywhere of:

$$b_n \to b, \qquad f_n \to f, \qquad y_{0,n} \to y_0, \qquad y_{l,n} \to y_l, \qquad y_{r,n} \to y_r.$$

Thanks to the dominated convergence theorem and to the limit uniqueness, we see that y satisfies (50) and (52) almost everywhere, and this provides (54). □

A.4. Uniqueness of the weak solution

We have proved the existence of a weak solution to (45) and we have the bound (54), therefore the initial boundary value problem will be well posed once we have shown the uniqueness of the weak solution.

Proposition A.7. *Under the hypothesis of Theorem* 6*, there is only one weak solution to* (45)*.*

Proof. By linearity we only need to prove the uniqueness for $f=0$, $y_0=0$, $y_l=0$, $y_r=0$. Which is $\forall y \in L^\infty(\Omega_T)$:

$$\left(\forall \phi \in Adm(\Omega_T) \int_{\Omega_T} y.\big(\partial_t \phi + a.\partial_x \phi + (b+\partial_x a).\phi\big)\,dx\,dt = 0\right) \quad \Rightarrow \quad y = 0 \quad \text{a.e.}$$

Let y be such as above, we take:

- $y_n \in \mathcal{C}_c^1(\Omega_T \setminus P)$ such that $\|y_n - y\|_{L^2(\Omega_T)} \to 0$ and $\|y_n\|_{L^\infty(\Omega_T)}$ is bounded,
- $d_n \in \mathcal{C}^1(\Omega_T)$ such that $\|d_n - (b+\partial_x a)\|_{L^2(\Omega_T)} \to 0$ and $\|d_n\|_{L^\infty(\Omega_T)}$ is bounded.

We want $\phi_n \in Adm(\Omega_T)$ to be a strong solution of $\partial_t \phi_n + a.\partial_x \phi_n + d_n.\phi_n = y_n$, but the boundary conditions for functions in $Adm(\Omega_T)$ makes it a backward problem. Indeed for ϕ_n to be a test function we must have $\forall x \in [0,1]$, $\phi_n(T,x)=0$, $\forall t \in [0,T] \setminus \Gamma_l$, $\phi_n(t,0)=0$ and $\forall t \in [0,T] \setminus \Gamma_r$, $\phi_n(t,1)=0$. As we said previously the change of variables $t \to T-t$ transforms a backward problem in a regular forward one, which we can solve thanks to Section A.2. We just need to realize that the change of variables $t \to T-t$ sends the old P on the new P, the old $[0,T] \setminus \Gamma_l$ on the new $\Gamma_l \cup P_l$ and the old $[0,T] \setminus \Gamma_r$ on the new $\Gamma_l \cup P_r$.

$$\text{And therefore:} \quad \forall n \in \mathbb{N}, \quad \int_{\Omega_T} y.\big(y_n + \phi_n(b+\partial_x a - d_n)\big)\,dx\,dt = 0.$$

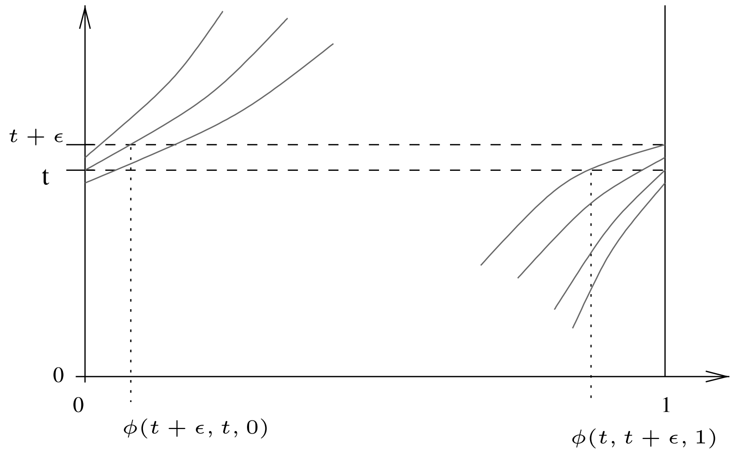
Now thanks to the hypothesis on y_n and d_n, and to (51), when $n \to +\infty$ we get $\int_{\Omega_T} |y(t,x)|^2\,dx\,dt = 0$. $\square$

A.5. Additional properties of y

Until now weak solutions had only the L^∞ regularity but in fact we have more.

Lemma 4. *If a and $\partial_x a$ are continuous and if the sets $P_l = \{t \in [0,T] \mid a(t,0)=0\}$ and $P_r = \{t \in [0,T] \mid a(t,1)=0\}$ have a finite number of connected components, and if b and f are in $L^\infty(\Omega_T)$ then $\forall p < +\infty$ we have $\|y\|_{L^p(0,1)} \in \mathcal{C}^0([0,T])$.*

Proof. Let $t \in [0,T]$ and $\epsilon \geqslant 0$. Reducing ϵ if necessary we can suppose that $a(s,0)$ and $a(s,1)$ have a constant sign on $[t,t+\epsilon]$. Hence we will prove the result in the case $a(t,0) \geqslant 0$ and $a(t,1) \geqslant 0$ (the other cases being similar). This implies $h(t,0) \geqslant t+\epsilon$ and $e(t+\epsilon,1) \leqslant t$:



Now we have:

$$\|y(t+\epsilon,.)\|^p_{L^p(0,1)} = \int_0^{\phi(t+\epsilon,t,0)} \left|y(t+\epsilon,x)\right|^p dx + \int_{\phi(t+\epsilon,t,0)}^1 \left|y(t+\epsilon,x)\right|^p dx$$

since $\phi(t+\epsilon,t,0) \underset{\epsilon\to 0}{\longrightarrow} 0$ and $y \in L^\infty(\Omega_T)$ the first integral tends to 0. Then, if $x \in [\phi(t+\epsilon,t,0),1]$ we recall that thanks to (52) and after performing the change of variables $\tilde{x} = \phi(t,t+\epsilon,x)$ one has:

$$\begin{aligned}\int_{\phi(t+\epsilon,t,0)}^1 \left|y(t+\epsilon,x)\right|^p dx = \int_0^{\phi(t,t+\epsilon,1)} \Bigg| & y(t,\tilde{x}).\exp\left(\int_t^{t+\epsilon} b\big(s,\phi(s,t,\tilde{x})\big)\,ds\right) \\ & + \int_t^{t+\epsilon} f\big(t,\phi(r,t,\tilde{x})\big).\exp\left(\int_r^{t+\epsilon} b\big(r',\phi(r',t,\tilde{x})\big)\,dr'\right) dr\Bigg|^p \\ & \times \exp\left(\int_t^{t+\epsilon} \partial_x a\big(s,\phi(s,t,\tilde{x})\big)\,ds\right) d\tilde{x}. \end{aligned} \tag{55}$$

And finally since $\phi(t,t+\epsilon,1) \underset{\epsilon\to 0^+}{\longrightarrow} 1$, $f, b, y \in L^\infty(\Omega_T)$ and $\partial_x a \in \mathcal{C}^0(\Omega_T)$ we get

$$\int_{\phi(t+\epsilon,t,0)}^1 \left|y(t+\epsilon,x)\right|^p dx \underset{\epsilon\to 0^+}{\longrightarrow} \int_0^1 \left|y(t,x)\right|^p dx.$$

The other geometries of the characteristics are treated in the same way. And the argument is clearly reversible in time so we also have the case $\epsilon \leqslant 0$. □

Now we can get some additional regularity for y.

Proposition A.8. *If a and $\partial_x a$ are continuous, if the sets $P_l = \{t \in [0,T] \mid a(t,0)=0\}$ and $P_r = \{t \in [0,T] \mid a(t,1)=0\}$ have a finite number of connected components, if y_0, y_l y_r are essentially bounded and if b and f are in $L^\infty(\Omega_T)$ then $\forall p < +\infty$ we have $y \in \mathcal{C}^0([0,T], L^p(0,1))$.*

Proof. We take $t=0$ and $\epsilon > 0$. Reducing ϵ if necessary, we can suppose that $a(s,0)$ and $a(s,1)$ have a constant sign on $[t, t+\epsilon]$. We will prove the result in the case $a(t,0) \geqslant 0$ and $a(t,1) \leqslant 0$ (the others can be treated in the same way). This implies $h(0,1), h(0,0) \geqslant \epsilon$.

Let $\gamma > 0$, since $y_0 \in L^\infty(0,1)$ we have a function $\tilde{y}_0 \in \mathcal{C}^0([0,1])$ such that $\|y_0 - \tilde{y}_0\|_{L^p(0,1)} \leqslant \gamma$. We now consider $\tilde{y}$ the weak solution of (45) with boundary value y_l and y_r and initial value $\tilde{y}_0$. Now by linearity it is clear that $y - \tilde{y}$ is solution to (45) with boundary value 0 and initial value $y_0 - \tilde{y}_0$. Therefore the previous lemma asserts that $\|y(t,.) - \tilde{y}(t,.)\|_{L^p(0,1)}$ is continuous and we see that for t sufficiently small $\|y(t,.) - \tilde{y}(t,.)\|_{L^p(0,1)} \leqslant 2.\gamma$.

Now since $\tilde{y}$ satisfies (52), since $b, f, \tilde{y} \in L^\infty(\Omega_T)$ and more importantly since $\tilde{y}_0$ continuous, we obtain $\tilde{y}(\epsilon, x) \underset{\epsilon \to 0^+}{\longrightarrow} \tilde{y}_0(x)$ for any x in $(0,1)$, therefore we can conclude that $\int_{\phi(\epsilon,0,0)}^{\phi(\epsilon,0,1)} |\tilde{y}(\epsilon,x) - \tilde{y}_0(x)|^p\,dx \underset{\epsilon \to 0^+}{\longrightarrow} 0$. And finally we conclude that for ϵ sufficiently small $\|\tilde{y}(\epsilon,.) - \tilde{y}_0(.)\|_{L^p(0,1)} \leqslant \gamma$, which implies that for ϵ small enough:

$$\big\| y(\epsilon,.) - y_0(.) \big\|_{L^p(0,1)} \leqslant 4\gamma.$$

We can both translate and reversen the argument in time. □

To finish this part we will prove an inequality about the continuity property of the linear operator providing y in term of f, y_0, y_l and y_r.

Proposition A.9. *If a and $\partial_x a$ are continuous and if the sets $P_l = \{t \in [0,T] \mid a(t,0)=0\}$ and $P_r = \{t \in [0,T] \mid a(t,1)=0\}$ have a finite number of connected components then we have the inequality*:

$$\begin{aligned} &\forall t \in [0,T] \\ &\quad \big\| y(t,.) \big\|_{L^1(0,1)} \leqslant \big(\|f\|_{L^1((0,t)\times(0,1))} + \|y_0\|_{L^1(0,1)} + \|y_l\|_{L^1((0,t)\cap\Gamma_l)} + \|y_r\|_{L^1((0,t)\cap\Gamma_r)} \big) \\ &\qquad\qquad \times \|a\|_{L^\infty(\Omega_T)}\big).e^{t(\|\partial_x a\|_{L^\infty(\Omega_T)} + \|b\|_{L^\infty(\Omega_T)})}. \end{aligned} \tag{56}$$

Proof. Let us first suppose that $a(s,0), a(s,1) \geqslant 0$ on $[0,T]$, this implies $h(0,0) \geqslant t$ and $e(t,1) = 0$, therefore we can write:

$$\big\| y(t,.) \big\|_{L^1(0,1)} \leqslant \int_0^{\phi(t,0,0)} \left| y_l\big(e(t,x)\big).\exp\left(\int_{e(t,x)}^t b\big(r, \phi(r,t,x)\big)\,dr \right) \right| dx \tag{57}$$

$$+ \int_0^{\phi(t,0,0)} \left| \int_{e(t,x)}^t f\big(t, \phi(r,t,x)\big).\exp\left(\int_s^t b\big(s, \phi(s,t,x)\big)\,ds \right) dr \right| dx \tag{58}$$

$$+\int_{\phi(t,0,0)}^{1}\left|y_0\big(\phi(0,t,x)\big).\exp\left(\int_0^t b\big(r,\phi(r,t,x)\big)\,dr\right)\right|dx \tag{59}$$

$$+\int_{\phi(t,0,0)}^{1}\left|\int_0^t f\big(t,\phi(r,t,x)\big).\exp\left(\int_s^t b\big(s,\phi(s,t,x)\big)\,ds\right)dr\right|dx \tag{60}$$

$$= I_1+I_2+I_3+I_4. \tag{61}$$

Now we will treat each I_k separately. In I_1 we perform the change of variables: $s=e(t,x)$ (or equivalently $x=\phi(t,s,0)$) and we get:

$$I_1=\int_0^t \big|y_l(s)\big|.a(s,0).\exp\left(\int_s^t b(r,\phi(r,s,0))+\partial_x a\big(r,\phi(r,s,0)\big)\,dr\right)ds.$$

Therefore we have $I_1 \leqslant \|y_l\|_{L^1(0,t)}.\|a\|_{L^\infty(\Omega_T)}.e^{t.(\|\partial_x a\|_{L^\infty(\Omega_T)}+\|b\|_{L^\infty(\Omega_T)})}$.

For the second integral we have:

$$I_2=\int_0^{\phi(t,0,0)}\left|\int_{e(t,x)}^t f\big(t,\phi(r,t,x)\big).\exp\left(\int_s^t b\big(s,\phi(s,t,x)\big)\,ds\right)dr\right|dx \tag{62}$$

$$\leqslant \int_0^t\int_{\phi(s,0,0)}^{\phi(t,0,0)} \big|f\big(t,\phi(r,t,x)\big)\big|.\exp\left(\int_s^t b\big(s,\phi(s,t,x)\big)\,ds\right)dx\,dr. \tag{63}$$

This time we perform the change of variables: $\tilde{x}=\phi(r,t,x)$. And we get:

$$I_2 \leqslant e^{t(\|\partial_x a\|_{L^\infty(\Omega_T)}+\|b\|_{L^\infty(\Omega_T)})}\times\int_0^t\int_0^{\phi(r,0,0)}\big|f(t,\tilde{x})\big|\,d\tilde{x}\,dr. \tag{64}$$

In the same way we obtain:

$$I_3 \leqslant e^{t(\|b\|_{L^\infty(\Omega_T)}+\|\partial_x a\|_{L^\infty(\Omega_T)})}\times\int_0^{\phi(0,t,1)}\big|y_0(\tilde{x})\big|\,d\tilde{x}.$$

And finally for I_4 we use $\tilde{x}=\phi(r,t,x)$ to obtain:

$$I_4 \leqslant e^{t(\|b\|_{L^\infty(\Omega_T)}+\|\partial_x a\|_{L^\infty(\Omega_T)})}\int_0^t\int_{\phi(r,0,0)}^{\phi(r,t,1)}\big|f(t,\tilde{x})\big|\,d\tilde{x}\,dr.$$

Combining the inequalities on I_1, I_2, I_3 and I_4 we get (56). However we supposed that $a(s,0)$ and $a(s,1)$ did not change signs between on $[0,T]$. Therefore if either $a(s,0)$ or $a(s,1)$ change sign at time t_1 we only have the desired estimates separately on $[t_0,t_1]$ and on $[t_1,t_2]$ where on each interval, $a(s,0)$ and $a(s,1)$ do not change sign. More precisely if $t \in [t_1,t_2]$ we have:

$$\begin{aligned}\|y(t_1,.)\|_{L^1(0,1)} &\leqslant \big(\|f\|_{L^1((t_0,t_1)\times(0,1))} + \|y(t_0,.)\|_{L^1(0,1)} + \|y_l\|_{L^1((t_0,t_1)\cap\Gamma_l)} + \|y_r\|_{L^1((t_0,t_1)\cap\Gamma_r)}\big) \\ &\quad \times \|a\|_{L^\infty(\Omega_T)}\big).e^{(t_1-t_0)(\|\partial_x a\|_{L^\infty(\Omega_T)}+\|b\|_{L^\infty(\Omega_T)})}, \\ \|y(t,.)\|_{L^1(0,1)} &\leqslant \big(\|f\|_{L^1((t_1,t)\times(0,1))} + \|y(t_1,.)\|_{L^1(0,1)} + \|y_l\|_{L^1((t_1,t)\cap\Gamma_l)} + \|y_r\|_{L^1((t_1,t)\cap\Gamma_r)}\big) \\ &\quad \times \|a\|_{L^\infty(\Omega_T)}\big).e^{(t-t_1)(\|\partial_x a\|_{L^\infty(\Omega_T)}+\|b\|_{L^\infty(\Omega_T)})}.\end{aligned}$$

And now we can substitute $\|y(t_1,.)\|_{L^1(0,1)}$ in the right side of (65) with the right side of (65), which provides (56) on the whole interval $[t_0,t_2]$. Finally since we know that $a(s,0)$ and $a(s,1)$ change sign only a finite number of time, the previous argument allows us to extend (56) to $[0,T]$. □

Remark 10. The previous estimate and the well posedness in $L^\infty(\Omega_T)$ of the initial boundary value problem (11) for data y_0, y_l, y_r and f in L^∞ show that the same problem is well posed in $\mathcal{C}([0,T];L^1(0,1))$ with data in L^1. And then since the equation is linear and because we have both the well-posedness in $L^\infty(\Omega_T)$ with essentially bounded data, and also the well-posedness in $\mathcal{C}^0([0,T];L^1(0,1))$ with summable data we can interpolate the two results and get well posedness in $\mathcal{C}^0([0,T];L^p(0,1))$ with data in L^p.

References

[1] A. Bressan, A. Constantin, Global conservative solutions of the Camassa–Holm equation, Arch. Ration. Mech. Anal. 183 (2) (2007) 215–239.

[2] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (11) (1993) 1661–1664.

[3] G.M. Coclite, H. Holden, K.H. Karlsen, Global weak solutions to a generalized hyperelastic-rod wave equation, SIAM J. Math. Anal. 37 (4) (2005) 1044–1069.

[4] A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 26 (2) (1998) 303–328.

[5] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Math. 181 (2) (1998) 229–243.

[6] A. Constantin, J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, Math. Z. 233 (1) (2000) 75–91.

[7] A. Constantin, V.S. Gerdjikov, R.I. Ivanov, Inverse scattering transform for the Camassa–Holm equation, Inverse Problems 22 (6) (2006) 2197–2207.

[8] A. Constantin, L. Molinet, Global weak solutions for a shallow water equation, Comm. Math. Phys. 211 (1) (2000) 45–61.

[9] A. Constantin, W. Strauss, Stability of peakons, Comm. Pure Appl. Math. 53 (2000) 603–610.

[10] J.-M. Coron, Global Asymptotic Stabilization for controllable systems without drift, Math. Control Signals Systems 5 (1992) 295–312.

[11] J.-M. Coron, On the null asymptotic stabilization of 2-D incompressible Euler equation in a simply connected domain, SIAM J. Control Optim. 37 (6) (1999) 1874–1896.

[12] H.-H. Dai, Model equations for nonlinear dispersive waves in a compressible Mooney–Rivlin rod, Acta Mech. 127 (1998) 193–207.

[13] R. Danchin, A few remarks on the Camassa–Holm equation, Differential Integral Equations 14 (8) (2001) 953–988.

[14] R. Danchin, A note on well-posedness for Camassa–Holm equation, J. Differential Equations 192 (2) (2003) 429–444.
[15] J. Escher, Z. Yin, Initial boundary value problems of the Camassa–Holm equation, Comm. Partial Differential Equations 33 (1–3) (2008) 377–395.
[16] J. Escher, Z. Yin, Initial boundary value problems for nonlinear dispersive wave equations, J. Funct. Anal. 256 (2) (2009) 479–508.
[17] A.S. Fokas, B. Fuchssteiner, Symplectic structures, their Bäcklund transformations and hereditary symmetries, Phys. D 4 (1) (1981/1982) 47–66.
[18] O. Glass, Asymptotic stabilizability by stationary feedback of the two-dimensional Euler equation: the multiconnected case, SIAM J. Control Optim. 44 (3) (2005) 1105–1147.
[19] O. Glass, Controllability and asymptotic stabilization of the Camassa–Holm equation, J. Differential Equations 245 (6) (2008) 1584–1615.
[20] A. Himonas, G. Misiolek, Well posedness of the Cauchy problem for a shallow water equation on the circle, J. Differential Equations 162 (2000) 479–495.
[21] R.S. Johnson, Camassa–Holm, Korteweg–de Vries and related models for water waves, J. Fluid Mech. 455 (2002) 63–82.
[22] A.V. Kazhikhov, Note on the formulation of the problem of flow through a bounded region using equations of perfect fluid, J. Appl. Math. Mech. 44 (5) (1980) 947–950 (in Russian).
[23] J. Lenells, Traveling wave solutions of the Camassa–Holm equation, J. Differential Equations 217 (2) (2005) 393–430.
[24] Y.A. Li, P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Differential Equations 162 (1) (2000) 27–63.
[25] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge Texts Appl. Math., vol. 27, Cambridge University Press, Cambridge, 2002, ISBN: 0-521-63057-6; 0-521-63948-4.
[26] Z. Xin, P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math. 53 (11) (2000) 1411–1433.
[27] V.I. Yudovich, A two dimensional problem of unsteady flow of an ideal incompressible fluid across a given domain, Mat. Sb. 31 (1840) (1964) 562–588 and Mat. Sb. 64 (106) (1964) 277–304.
[28] G. Zhang, Z. Qiao, F. Liu, Cusp and smooth solitons of the Camassa–Holm equation under an inhomogeneous boundary condition, Pacific J. Appl. Math. 1 (1) (2008) 105–121.